NILPOTENT PERMUTATIONAL WREATH PRODUCTS

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PREFACE

Unless otherwise indicated, results in this thesis are my own.

The thesis has been composed by myself.

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ABSTRACT

D.Shield gave a formula in Bull.Austral.Math.Soc. 17 (1977) for the nilpotency class of the nilpotent standard wreath product $A \text{ wr } B$. This thesis provides results towards a generalisation, if it exists, of the formula to the nilpotent permutational wreath product $A \text{ wr}^{\Lambda} B$, where A and B are groups and B acts on the set Λ.

Let P_r be the Sylow p-subgroup of the symmetric group on the set (p^r) of p^r symbols. For a nilpotent group G let $c(G)$ denote the nilpotency class of G. Thesis work of A.J.Scott (Edinburgh 1975) shows we may restrict attention to $A \text{ wr}^{\Lambda} B$ where A is a nilpotent p-group of finite exponent, $\Lambda = (p^r)$ and B is a transitive subgroup of P_r. We show $c(C_{p^n} \text{ wr}^{(p^r)} P_r) = p^{r-1}\{p + (p-1)(n-1)\}$, which by Shield's formula is $c(C_{p^n} \text{ wr } C_{p^r})$. Thus if the transitive subgroup B of P_2 has exponent p^2, then $c(C_{p^n} \text{ wr}^{(p^2)} B) = p\{p + (p-1)(n-1)\}$. If B is a transitive subgroup of P_2 of exponent p and order p^t, $2 \leq t \leq p$, then $c(C_{p^n} \text{ wr}^{(p^2)} B) = p + n(p-1)(t-1)$. We conjecture that if B is a transitive subgroup of P_r, and A is a nilpotent p-group of finite exponent, then $c(A \text{ wr}^{(p^r)} B) = \max\{ c(C_p \text{ wr}^{(p^r)} B)w + (p-1)d(B)(s(w) - 1) : 1 \leq w \leq c(A) \}$, where a commutator of weight w in A has order at most $p^{s(w)}$, and $d(B)$ is the cpp-class of B.

A.J.Weir gave the lower central structure of P_r in Proc.Am.

Math.Soc. <u>6</u> (1955) . We determine the lower central and cpp-structure of $C_{p^n} \text{ wr}^{(p^r)} P_r$ and discuss the lower central structure of $C_{p^n} \text{ wr } C_{p^r}$.

CONTENTS

" To know the world, one must construct it."

Pavese

INTRODUCTION

Let A and B be groups, and let B act on the set Λ. The aim of this thesis is to obtain the nilpotency class of the nilpotent permutational wreath products of the form $A \text{ wr}^{\Lambda} B$. In the special case where $\Lambda = B$ and B acts on itself by right multiplication we have the standard wreath product $A \text{ wr}^{B} B = A \text{ wr } B$. A formula for the nilpotency class of $A \text{ wr } B$ was obtained by Shield in [17] : see Theorem 2.2 below. We do not obtain a full generalisation of this remarkable result, but are able to give the class in certain cases, and suggest the general form such a formula - if it exists - might take. The ideas behind the proofs given are on the whole not particularly difficult, but unfortunately are often rather tedious to expound formally. Several diagrams are included in the hope of making the arguments easier to follow.

The four key sources of background for the thesis are :

1. Two papers by Shield , [16] and [17] , particularly [17] , which contains the result for the standard wreath product.

2. A paper by Liebeck , [9] , on the nilpotency of $C_{p^n} \text{ wr } B$ where B is a finite abelian p-group, which was used by Shield and is crucial in many of the results obtained below.

3. The thesis of Scott , [14] , which in Theorem 2.4 below gives conditions for $A \text{ wr}^{\Lambda} B$ to be nilpotent : we give a full

proof as the material has not been published.

4. A paper based on the thesis of Weir, [18], which gives details of the nilpotency structure of the Sylow p-subgroup P_r of the symmetric group on p^r symbols.

In Chapter I we give definitions of the various wreath products and the (a,b,e)-series of Shield, which include the lower central and cpp (commutator-p th-power) - series. We give some associated basic results, and define some important notation for the wreath product $P_r = \underbrace{C_p \text{ wr } C_p \text{ wr } \ldots \text{ wr } C_p}_{r\ C_p\text{'s}}$.

In Chapter II we state Shield's formula for the nilpotency class of the nilpotent wreath products A wr B. Note that the formula requires extensive knowledge of B : we require the cpp-class of B, and either the order of the terms in the cpp-series of B or a standard basis for B so that we can calculate the constant $a(B)$ used in the formula. With the present scant knowledge of the cpp-series of even fairly well-known groups, the formula provides the nilpotency class in principle only, and work is being done to obtain alternative formulations for special cases : see for example [12]. We state and prove the theorem of Scott mentioned above which gives necessary and sufficient conditions on A, B and Λ for $A \text{ wr}^{\Lambda} B$ to be nilpotent. This enables us to restrict attention to permutational wreath products $A \text{ wr}^{\Lambda} B$ where A is a nilpotent p-group of finite exponent and

B is a transitive subgroup of P_r, the Sylow p-subgroup of the symmetric group on the p^r symbols $(p^r) = \Lambda$. We give the important results of Liebeck referred to above, and generalise results of Neumann, [13], on the centre of a standard wreath product to the permutational wreath product.

Chapter III sees the first proof of the nilpotency class of $C_{p^n} \text{ wr}^{(p^r)} P_r$, and a proof of the cpp-class of $C_{p^n} \text{ wr}^{(p^r)} P_r$, both using Shield's results. It is rather surprising that both the nilpotency and cpp- class of $C_{p^n} \text{ wr}^{(p^r)} P_r$ are exactly those of a relatively small subgroup which is isomorphic to $C_{p^n} \text{ wr } C_{p^r}$. These provide upper bounds for the nilpotency and cpp- class of $C_{p^n} \text{ wr}^{(p^r)} B$, where B is a transitive subgroup of P_r, since $C_{p^n} \text{ wr}^{(p^r)} B$ is a subgroup of $C_{p^n} \text{ wr}^{(p^r)} P_r$. We conjecture in Conjecture 3.2 that if we know enough about the lower central structure of A, and if B is a transitive subgroup of P_r, then it is sufficient to find the nilpotency class of $C_{p^n} \text{ wr}^{(p^r)} B$ to determine the nilpotency class of $A \text{ wr}^{(p^r)} B$. We return to this conjecture in Chapter VII. Finally, we construct an important set of commutators $\{ \underline{g}_i \}$ of $C_{p^n} \text{ wr}^{(p^r)} P_r$, and present a plausibility argument for the nilpotency class of $C_{p^n} \text{ wr}^{(p^r)} P_r$ which does not use Shield's results.

In Chapter IV we generalise results of Weir, [18], on the nilpotency structure of P_r to $C_{p^n} \text{ wr}^{(p^r)} P_r$, for which we determine the lower central and cpp- structure, and give a basis for the terms in each series. This enables us to calculate the constant $a(C_{p^n}\ wr^{(p^r)}P_r)$. We discuss the lower central structure of $C_{p^n}\ wr\ C_{p^r}$.

In Chapter V we give a second proof of the nilpotency class of $C_{p^n}\ wr^{(p^r)}P_r$ which avoids using Shield's result.

In Chapter VI we give an example of the more general use of the methods of Chapter V , to calculate the nilpotency class of $C_{p^n}\ wr^{(p^2)}B$ where B is a transitive subgroup of P_2 of exponent p . Note that if B is a transitive subgroup of P_2 of exponent p^2 , then $C_{p^n}\ wr^{(p^2)}B$ contains a subgroup isomorphic to $C_{p^n}\ wr\ C_{p^2}$, which we saw has the same nilpotency class as $C_{p^n}\ wr^{(p^2)}P_2$; and so since $C_{p^n}\ wr^{(p^2)}B$ is sandwiched between the two, its class is that of $C_{p^n}\ wr^{(p^2)}P_2$. Thus we have completely determined the nilpotency class of the permutational wreath products $C_{p^n}\ wr^{(p^2)}B$ where B is a transitive subgroup of P_2 . We also show that there is one group to within isomorphism of the form $C_{p^n}\ wr^{(p^2)}B$ where B is transitive, has exponent p and order p^t , $2 \leqslant t \leqslant p$, and that B is of maximal class.

In Chapter VII we approach the problem of obtaining a formula for the nilpotency class of a permutational wreath product from the "top end" , and prove some results towards Conjecture 3.2 . We elaborate the ideas behind Shield's proof of the formula for the standard case, and generalise some of these to the permutational case. We are able to show that the truth of

Conjecture 3.2 depends solely on proving that the nilpotency class of $C_{p^n} \text{ wr}^{(p^r)} B$ rises in equal steps with n for $n \geq 1$. It seems likely that such a result would be obtained from more work on $C_p \text{ wr}^{(p^r)} B$: such a result for the case of the standard wreath product follows from work on the nilpotency class of $C_p \text{ wr } B$, and in particular from the standard basis obtained for B.

CHAPTER I : Notation, definitions and some basic results.

Notation

As usual, the natural numbers $0, 1, 2, \ldots$ will be denoted by $\mathbb{N}$, the integers by $\mathbb{Z}$, and the integers strictly greater than 0 by $\mathbb{Z}^+$.

The letter p will always denote a prime.

The set $\{1, 2, \ldots, p^r\}$ will sometimes be written (p^r).

Given a prime p,
τ will be the map from $\mathbb{Z}$ to $\{0, 1, \ldots, p-1\}$ such that
$\tau(k) \equiv k \mod p$ for all k in $\mathbb{Z}$, and
ν will be the map from $\mathbb{Z}$ to (p) such that
$\nu(k) \equiv k \mod p$ for all k in $\mathbb{Z}$.

The sign $<$ is to be interpreted as "strictly less than", and similarly for $>$.
For example, if G is a group then $H < G$ denotes that H is a proper subgroup of G.
Similarly, for a set Λ, $\Theta \subset \Lambda$ denotes that Θ is a proper subset of Λ.

$\lceil x \rceil$ is the smallest integer greater than or equal to x.
$\lfloor x \rfloor$ is the greatest integer less than or equal to x.

For n and k in $\mathbb{N}$ such that $k \leq n$ let $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

$|G|$ will denote the order of the group G, and $|\Lambda|$ will denote the cardinality of the set Λ .

If the elements $g_1, \ldots , g_n, \ldots$ of the group G generate G , we write $G = \langle g_1, \ldots , g_n, \ldots \rangle$.

C_k is the cyclic group of order k .

S_Λ is the symmetric group on the set Λ , i.e. the permutation group on $|\Lambda|$ points, or symbols.

For m in $\mathbb{N}$ and the group G we denote the subgroup of G generated by the m th powers of elements of G by G^m .

We will denote the identity of a group by 1 .

$C_G(H)$ is the centraliser in the group G of $H \leq G$.

Let Λ be a non-empty set and let the group H act on Λ, that is, there exists a homomorphism θ from H to S_Λ, and for h in H, the image of λ in Λ under the permutation $h\theta$ is denoted by λh. Note that for h_1, h_2 in H, λ in Λ,

$$\lambda(h_1h_2) = (\lambda h_1)h_2 .$$

The kernel of θ is the <u>stabiliser of Λ in H</u>, denoted by

$$St_H(\Lambda) = \{ h\in H : \lambda h = \lambda \quad \forall\lambda\in\Lambda \} .$$

Clearly H is a <u>permutation group</u> on Λ, i.e. a subgroup of S_Λ if and only if $St_H(\Lambda) = \langle 1 \rangle$.

We call "Λ, H" a <u>pair</u>.

"Λ, H" is a <u>transitive pair</u> if H acts transitively on Λ, so, given points λ_1 and λ_2 in Λ we can find h in H such that

$$\lambda_1 h = \lambda_2 .$$

Where the context is clear we may just say that H is transitive.

"Λ, H" is a <u>trivial pair</u> if $St_H(\Lambda) = H$, and
"Λ, H" is a <u>faithful pair</u> if $St_H(\Lambda) = \langle 1 \rangle$. Unless otherwise specified we take "Λ, $\{H/ St_H(\Lambda)\}$" to be the pair for which the action of $H/ St_H(\Lambda)$ on Λ is induced by the action of H on Λ

$$\text{i.e. } \lambda h\, St_H(\Lambda) = \lambda h \quad \forall\ \lambda\in\Lambda,\ h\in H .$$

Now let "Λ_1, H_1" and "Λ_2, H_2" be pairs, and assume Λ_1 and Λ_2 are disjoint.

Let $\Lambda_1\times\Lambda_2 = \{ (\lambda_1, \lambda_2) : \lambda_1\in\Lambda_1, \lambda_2\in\Lambda_2 \}$.

Let W be the set of all permutations w of $\Lambda_1\times\Lambda_2$ such that

$$(\lambda_1, \lambda_2)w = (\lambda_1 h_1(\lambda_2), \lambda_2 h_2)$$

where $h_1(\cdot) : \Delta_2 \longrightarrow H_1$ and $h_2 \in H_2$. Since $h_1(\lambda_2)$ and h_2 are permutations of Δ_1 and Δ_2 respectively, w is a permutation of $\Delta_1 \times \Delta_2$. Note that to determine the second coordinate of the image we need only λ_2, but that we require both λ_1 and λ_2 to determine the first coordinate of the image. Then w is determined by $|\Delta_2|$ permutations in H_1 and one permutation in H_2.

Let another permutation w' of $\Delta_1 \times \Delta_2$ be given by

$$(\lambda_1, \lambda_2)w' = (\lambda_1 h_1'(\lambda_2), \lambda_2 h_2') .$$

Then we define the product ww' by

$$\begin{aligned}(\lambda_1, \lambda_2)ww' &= ((\lambda_1, \lambda_2)w)w' ,\\ &= (\lambda_1 h_1(\lambda_2), \lambda_2 h_2)w' ,\\ &= (\lambda_1 h_1(\lambda_2)h_1'(\lambda_2 h_2), \lambda_2 h_2 h_2') . \qquad \text{.......(1)}\end{aligned}$$

Clearly $ww' \in W$, and ww' is the identity permutation when both

$$h_1'(\lambda_2) = (h_1(\lambda_2 h_2^{-1}))^{-1} \quad \text{and}$$

$$h_2' = h_2^{-1} .$$

Multiplication is associative, for if $x, y, z \in W$ then

$$\begin{aligned}(\lambda_1, \lambda_2)(xy)z &= ((\lambda_1, \lambda_2)(xy))z ,\\ &= (((\lambda_1, \lambda_2)x)y)z ,\\ &= ((\lambda_1, \lambda_2)x)(yz) ,\\ &= (\lambda_1, \lambda_2)x(yz) .\end{aligned}$$

Hence W is a group. We call W the <u>complete</u> or <u>unrestricted</u> <u>wreath product</u> of H_1 by H_2, and write

$$W = {}^{\Delta_1}H_1 \mathrm{Wr}^{\Delta_2}H_2 \quad , \text{ or}$$

$$W = H_1 \text{ Wr } H_2 \qquad \text{where no confusion will arise.}$$

Note that as an abstract group the definition of W is independent of Λ_1, i.e. for pairs "Λ_1, H_1", "Λ_1', H_1", "Λ_2, H_2",

$$^{\Lambda_1'}H_1 \text{Wr}^{\Lambda_2} H_2 \cong {}^{\Lambda_1}H_1 \text{Wr}^{\Lambda_2} H_2 ,$$

and so we may write $H_1 \text{Wr}^{\Lambda_2} H_2$ if we are just considering the abstract group.

The <u>standard unrestricted wreath product</u> $H_1 \text{Wr } H_2$ of two abstract groups H_1 and H_2 is defined as

$$H_1 \text{Wr}^{H_2} H_2$$

where the group H_2 acts by right multiplication on the set H_2, i.e. "H_2, H_2" is the right regular representation of H_2.

From the definition of a permutation w in W we see that we may write w as a formal product $h_1(\cdot)h_2$, where

$$(\lambda_1, \lambda_2)w = (\lambda_1 h_1(\lambda_2), \lambda_2 h_2) ,$$

and we define $(h_1 h_1')(\cdot)$ by

$$(h_1 h_1')(\cdot) = h_1(\cdot)h_1'(\cdot) \qquad \forall \; h_1(\cdot), h_1'(\cdot) : \Lambda_2 \longrightarrow H_1 .$$

The set of maps $h_1(\cdot)$ from Λ_2 to H_1 with the above multiplication is the <u>cartesian power $\text{Cr } H_1{}^{\Lambda_2}$</u> of H_1, and is a group with identity $1(\cdot)$ given by

$$1(\lambda_2) = 1 \qquad \forall \; \lambda_2 \in \Lambda_2 .$$

In W we will identify $h_1(\cdot)$ in $\text{Cr } H_1{}^{\Lambda_2}$ with the formal product $h_1(\cdot)1$, thus embedding $\text{Cr } H_1{}^{\Lambda_2}$ in W. Likewise we embed H_2 in W by identifying h_2 in H_2 with the formal product $1(\cdot)h_2$. Then

$$h_2^{-1}h_1(\cdot)h_2 = (1(\cdot)h_2^{-1})(h_1(\cdot)h_2)$$

$$= h_1(\cdot\, h_2^{-1}) \qquad \text{from (1)} .$$

Thus $\mathrm{Cr}\, H_1^{\Delta_2} \lhd W$. Clearly $W/\mathrm{Cr}\, H_1^{\Delta_2} \cong H_2$ and $\mathrm{Cr}\, H_1^{\Delta_2} \cap H_2$ is equal to $\langle 1 \rangle$. Hence W is the split extension or semi-direct product of $\mathrm{Cr}\, H_1^{\Delta_2}$ by H_2.

We call H_1 the <u>bottom</u> (or passive) <u>group</u> of W,

H_2 the <u>top</u> (or active) <u>group</u> of W,

and $\mathrm{Cr}\, H_1^{\Delta_2}$ the <u>base group, D,</u> of W.

Let f be a map from a set Δ to a group H. Then the <u>support of f</u> is defined by

$$\sigma(f) = \{ \lambda \in \Delta : f(\lambda) \neq 1 \} ,$$

and f is said to have <u>finite support</u> if $\sigma(f)$ is a finite set.

We now define the <u>λth coordinate subgroup</u> $H_{1,\lambda}$ of $\mathrm{Cr}\, H_1^{\Delta_2}$ for λ in Δ_2 by

$$H_{1,\lambda} = \{ h_1(\cdot) \in \mathrm{Cr}\, H_1^{\Delta_2} : \sigma(h_1(\cdot)) \subseteq \{\lambda\} \} .$$

The subgroup $H_{1,\lambda}$ is isomorphic to H_1 for each λ in Δ_2', and $\mathrm{Cr}\, H_1^{\Delta_2}$ is the cartesian product of its coordinate subgroups since the coordinate subgroups commute elementwise.

We define the <u>direct power $\mathrm{Dr}\, H_1^{\Delta_2}$</u> of H_1 as the subset of $\mathrm{Cr}\, H_1^{\Delta_2}$ consisting of functions with finite support. Then $\mathrm{Dr}\, H_1^{\Delta_2}$ is a group, which is normal in W. We will write $H_1^{\Delta_2}$ where ambiguity will not arise. Note $H_1^{\Delta_2}$ is the direct product of the coordinate subgroups $\{ H_{1,\lambda} : \lambda \in \Lambda_2 \}$ of $\mathrm{Cr}\ H_1^{\Lambda_2}$. Furthermore, each $h_1(\cdot)$ in $\mathrm{Dr}\ H_1^{\Lambda_2}$ may be written uniquely in the form

$$h_1(\cdot) = \prod_{\lambda\in\Lambda_2} f_\lambda$$

where $f_\lambda \in H_{1,\lambda}$. This product is well-defined since all but a finite number of f_λ's are trivial. We call f_λ the <u>λth factor</u> of $h_1(\cdot)$.

The <u>restricted wreath product</u> ${}^{\Lambda_1}H_1 \mathrm{wr}^{\Lambda_2} H_2$ of H_1 by H_2 is defined as the split extension of $\mathrm{Dr}\ H_1^{\Lambda_2}$ by H_2. The bottom, top, and base groups are defined analogously to those of the unrestricted wreath product, as H_1, H_2 and $\mathrm{Dr}\ H_1^{\Lambda_2}$ respectively.

The <u>standard restricted wreath product</u> $H_1 \mathrm{wr}\ H_2$ of two abstract groups H_1 and H_2 is defined as

$$H_1 \mathrm{wr}^{H_2} H_2$$

where H_2 acts by right multiplication on the set H_2.

A subgroup of the wreath product of some importance is the <u>diagonal subgroup</u>. Let A be a group and let Λ be a set. Then the <u>diagonal of $\mathrm{Cr}\ A^\Lambda$</u> is the set of all constant functions in $\mathrm{Cr}\ A^\Lambda$:

$$\mathcal{D}_{\mathrm{Cr}\ A^\Lambda} = \{ f \in \mathrm{Cr}\ A^\Lambda : f(\lambda) = f(\lambda') \quad \forall\ \lambda, \lambda' \in \Lambda \} .$$

The <u>diagonal of $\mathrm{Dr}\ A^\Lambda$</u> is the set of all constant functions in $\mathrm{Dr}\ A^\Lambda$:

$$\mathcal{D}_{\mathrm{Dr}\ A^\Lambda} = \{ f \in \mathrm{Dr}\ A^\Lambda : f(\lambda) = f(\lambda') \quad \forall\ \lambda, \lambda' \in \Lambda \} .$$

Clearly if Λ is infinite then $\mathcal{D}_{\mathrm{Dr}\,A^{\Lambda}}$ is trivial, since the support of f in $\mathrm{Dr}\,A^{\Lambda}$ is finite. It is easy to see that the diagonal is a group, and is normal in $\mathrm{Cr}\,A^{\Lambda}$ if and only if A is abelian.

The definition of the unrestricted wreath product is sufficient for constructing a wreath product W of finitely many pairs "Λ_1, H_1", "Λ_2, H_2", ... , "Λ_n, H_n" by

$$W = ((\cdots(({}^{\Lambda_1}H_1 \mathrm{Wr}^{\Lambda_2}H_2)\mathrm{Wr}^{\Lambda_3}H_3) \ \ldots\ )\mathrm{Wr}^{\Lambda_n}H_n) .$$

A most important property of this <u>iterated wreath product</u> is associativity, which enables us to remove the bracketing. We follow § 7.5 of [8] :

$${}^{\Lambda_1\times\Lambda_2}({}^{\Lambda_1}H_1\mathrm{Wr}^{\Lambda_2}H_2)\mathrm{Wr}^{\Lambda_3}H_3$$

is the set of all permutations $\underline{w}$ on $(\Lambda_1\times\Lambda_2)\times\Lambda_3$ of the following kind :

$$((\lambda_1, \lambda_2), \lambda_3)\underline{w} = ((\lambda_1, \lambda_2)w(\lambda_3), \lambda_3 h_3) ,$$
$$= ((\lambda_1 h_1(\lambda_2, \lambda_3), \lambda_2 h_2(\lambda_3)), \lambda_3 h_3) ,$$

where $w(\cdot) : \Lambda_3 \longrightarrow {}^{\Lambda_1}H_1\mathrm{Wr}^{\Lambda_2}H_2$,

$h_1(\cdot,\cdot) : \Lambda_2\times\Lambda_3 \longrightarrow H_1$,

$h_2(\cdot) : \Lambda_3 \longrightarrow H_2$, and

$h_3 \in H_3$.

$${}^{\Lambda_1}H_1\mathrm{Wr}^{(\Lambda_2\times\Lambda_3)}({}^{\Lambda_2}H_2\mathrm{Wr}^{\Lambda_3}H_3)$$

is the set of all permutations $\underline{w}'$ on $\Lambda_1\times(\Lambda_2\times\Lambda_3)$ such that

$$(\lambda_1, (\lambda_2, \lambda_3))\underline{w}' = (\lambda_1 h_1(\lambda_2, \lambda_3), (\lambda_2, \lambda_3)w')$$
$$= (\lambda_1 h_1(\lambda_2, \lambda_3), (\lambda_2 h_2(\lambda_3), \lambda_3 h_3))$$

where $w' \in {}^{\Delta_2}H_2 \mathrm{Wr} {}^{\Delta_3}H_3$ and $h_1(\cdot,\cdot)$, $h_2(\cdot)$ and h_3 are as above.

If we identify the sets $(\Delta_1\times\Delta_2)\times\Delta_3$ and $\Delta_1\times(\Delta_2\times\Delta_3)$ we find

$${}^{\Delta_1\times\Delta_2}({}^{\Delta_1}H_1 \mathrm{Wr} {}^{\Delta_2}H_2)\mathrm{Wr}{}^{\Delta_3}H_3 = {}^{\Delta_1}H_1 \mathrm{Wr}^{(\Delta_2\times\Delta_3)}({}^{\Delta_2}H_2 \mathrm{Wr}{}^{\Delta_3}H_3) . \qquad \square$$

<u>Note</u> We may omit the sets $\Delta_1, \Delta_2, \Delta_3, \ldots$ when from the context it is clear which permutational wreath product is being considered.

The <u>restricted iterated wreath product</u> of n pairs is defined as above, except for $i=1,\ldots,n-1$ the maps $h_i(\cdot, \ldots ,\cdot)$ from $\Delta_{i+1}\times \ldots \times\Delta_n$ to H_i have finite support.

The standard wreath product is not associative, as we shall see in a moment.

We will denote the standard wreath product of r cyclic groups of order a fixed prime p by

$$P_r = ((\cdots(\underbrace{C_p \mathrm{wr}\ C_p)\mathrm{wr} \ldots)\mathrm{wr}\ C_p}_{r\ C_p\text{'s}}) = C_p \mathrm{wr}\ C_p \mathrm{wr} \ldots \mathrm{wr}\ C_p$$

This is the Sylow p-subgroup of the symmetric group on (p^r), a proof of which may be found in § 7.5 of [8] .

Regarded as a permutational wreath product of r pairs "C_p , C_p" , P_r is an associative product, for example,

$$P_r \equiv P_{r-1}\mathrm{wr}\ C_p \equiv C_p \mathrm{wr}^{(p^{r-1})}P_{r-1} ,$$

but is not an associative product as a standard wreath product:

we show this for $r = 3$.

$$P_3 = (C_p \mathrm{wr}\ C_p)\mathrm{wr}\ C_p = C_p \mathrm{wr}^{(C_p \times C_p)}(C_p \mathrm{wr}\ C_p)$$

which has order $|C_p \mathrm{wr}\ C_p|^{|C_p|}|C_p| = (p^{p+1})^p.p$,

$$= p^{p^2+p+1} ,$$

$$= p^{p^2}.p^{p+1} ,$$

$$= |C_p|^{|C_p \times C_p|}|C_p \mathrm{wr}\ C_p| .$$

The standard wreath product

$$C_p \mathrm{wr}(C_p \mathrm{wr}\ C_p) = C_p \mathrm{wr}^{(C_p \mathrm{wr}\ C_p)}(C_p \mathrm{wr}\ C_p)$$

has order $|C_p|^{|C_p \mathrm{wr}\ C_p|}|C_p \mathrm{wr}\ C_p| = p^{p^{p+1}}.p^{p+1}$,

$$= p^{p^{p+1}+p+1} ,$$

$$\neq p^{p^2+p+1} .$$

We now give P_r in greater detail, both for later use and to illustrate the wreath product construction.

The group P_r acts on the set of symbols (p^r) . Any symbol λ in (p^r) may be written uniquely in the form

$$\lambda = a_o + a_1 p + a_2 p^2 + \dots + a_{r-1}p^{r-1} \qquad \text{.......(2)}$$

where $a_o \in \{1,2, \dots ,p\}$ and

$a_i \in \{0,1, \dots ,p-1\}$ for $1 \leq i \leq r-1$.

We may also regard (p^r) as the product of r sets of p symbols,

$$(p^r) = \Delta_1 \times \Delta_2 \times \dots \times \Delta_r$$

where $\Delta_1 = \{1,2, \dots ,p\}$ and $\Delta_i = \{0,1, \dots ,p-1\}$, for $2 \leq i \leq r$. In this notation, λ above is written in the form $(a_0, a_1, \ldots, a_{r-1})$. ....(3)

The group P_r is generated by the permutations $y_1, y_2, \ldots, y_r$ where for $i=1,2, \ldots, r$,

$$y_i = \prod_{j=1}^{p^{i-1}} (\, j,\ p^{i-1}+j,\ 2p^{i-1}+j,\ \ldots,\ p^i-p^{i-1}+j\,) .$$

Each y_i is of order p. Furthermore, y_i permutes the symbols (p^i) and leaves $\{p^i+1, \ldots, p^r\}$ fixed. Within (p^i), y_i shifts each symbol up p^{i-1} modulo p^i, i.e. for $\lambda = a_0 + a_1 p + \ldots + a_{r-1}p^{r-1}$,

$$\lambda y_1 = \begin{cases} \nu(a_0+1) + a_1 p + \ldots + a_{r-1}p^{r-1} & \text{if } a_1 = a_2 = \ldots = a_{r-1} = 0, \\ \lambda & \text{otherwise}, \end{cases} \qquad \ldots\ldots.(4)$$

and for $2 \leq i \leq r$,

$$\lambda y_i = \begin{cases} a_0 + a_1 p + \ldots + \tau(a_{i-1}+1)p^{i-1} + a_i p^i + \ldots + a_{r-1}p^{r-1} & \text{if } a_i = \ldots = a_{r-1} = 0, \\ \lambda & \text{otherwise}. \end{cases} \qquad \ldots\ldots.(5)$$

In the alternative notation for (p^r),

$$(a_0, a_1, \ldots, a_{r-1})y_1 = \begin{cases} (\nu(a_0+1), a_1, \ldots, a_{r-1}) & \text{if } a_1 = a_2 = \ldots = a_{r-1} = 0, \\ (a_0, a_1, \ldots, a_{r-1}) & \text{otherwise}, \end{cases} \qquad \ldots\ldots.(6)$$

and for $2 \leqslant i \leqslant r$,

$$(a_0, \dots, a_{r-1})y_i = \begin{cases} (a_0, \dots, \tau(a_{i-1}+1), a_i, \dots, a_{r-1}) \\ \qquad\qquad \text{if } a_i = \dots = a_{r-1} = 0 , \\ (a_0, \dots, a_{r-1}) \quad \text{otherwise} . \end{cases} \qquad \dots\dots(7)$$

In other words, y_i changes only the coordinate of Λ_i , and then only if the coordinates of $\Lambda_{i+1}, \dots, \Lambda_r$ are 0 , for $i=1,\dots,r$.

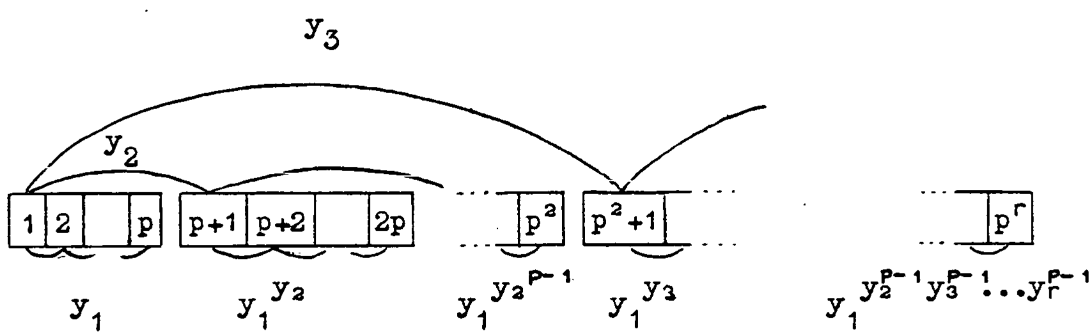


Fig. 1

In the simple case of $P_2 = C_p \mathrm{wr}\, C_p = \langle y_1, y_2 \rangle$, the bottom group is $\langle y_1 \rangle$, the top group is $\langle y_2 \rangle$ and the base group is generated by all conjugates of y_1 , i.e.

$$D = \langle y_1, y_1^{y_2}, y_1^{y_2^2}, \dots , y_1^{y_2^{p-1}} \rangle .$$

Now

$$\begin{aligned}(a_0, a_1)y_1^{y_2^k} &= (a_0, a_1)y_2^{-k}\, y_1\, y_2^{k} \\ &= (a_0, \tau(a_1-k))y_1 y_2^{k} .\end{aligned}$$

If $\tau(a_1-k) = 0$ then

$$(a_o, a_1)y_1^{y_2^k} = (\ \nu(a_o+1),\ \tau(a_1-k)\ )y_2^k\ ,$$
$$= (\ \nu(a_o+1),\ a_1\ )\ .$$

If $\tau(a_1-k) \neq 0$ then

$$(a_o, a_1)y_1^{y_2^k} = (\ a_o,\ \tau(a_1-k)\ )y_2^k\ ,$$
$$= (a_o,\ a_1)\ .$$

Hence $y_1^{y_2^k}$ changes the coordinate of Δ_1 if and only if the second coordinate , i.e. the coordinate of Δ_2 , is $k \bmod p$. In other words,

$$y_1^{y_2^k} = (\ kp+1,\ kp+2,\ \dots\ ,\ (k+1)p\ ) \quad \text{for } k=0,1,\dots,p-1\ .$$

Note that the conjugates of y_1 have mutually disjoint supports on (p^2), and so the conjugates commute. In fact, $\langle\ y_1^{y_2^k}\ \rangle$ is the k-th coordinate subgroup of the base group.

The diagonal subgroup of P_2 is just

$$\mathcal{D} = \langle\ y_1\ y_1^{y_2}\ y_1^{y_2^2} \dots\ y_1^{y_2^{p-1}}\ \rangle\ . \qquad \dots\dots\dots(8)$$

In the same way we find that y_1 has p^{r-1} conjugates in P_r , which are of the form

$$y_1^{y_2^{k_2} y_3^{k_3} \dots\ y_r^{k_r}} = (\ s+1,\ s+2,\ \dots\ ,\ s+p\ ) \qquad \dots\dots\dots(9)$$

where $s = k_r p^{r-1} + k_{r-1} p^{r-2} + \dots + k_2 p$,

and $0 \leqslant k_j \leqslant p-1$ for $j=2,\dots,r$.

There are no other conjugates, since every element in $\langle y_2, \dots, y_r \rangle$ must "shift" y_1 "up" by a multiple of p.

The base group of $P_r \left(\cong C_p \mathrm{wr}^{(p^{r-1})} P_{r-1}\right) = \langle y_1 \rangle \mathrm{wr}^{\Lambda} \langle y_2, \dots, y_r \rangle$, where $\Lambda = \Lambda_2 \times \Lambda_3 \times \dots \times \Lambda_r$, is thus generated by these p^{r-1} conjugates of y_1, and

$$\left\langle y_1^{y_2^{k_2} \dots y_r^{k_r}} \right\rangle \qquad \dots\dots\dots(10)$$

is the coordinate subgroup corresponding to $(k_2, k_3, \dots, k_r)$ of $\Lambda_2 \times \Lambda_3 \times \dots \times \Lambda_r$.

Similarly, for $P_r \left(\cong C_p \mathrm{wr}\ C_p \mathrm{wr}^{(p^{r-2})} P_{r-2}\right)$

$$= \langle y_1 \rangle \mathrm{wr} \langle y_2 \rangle \mathrm{wr}^{\Lambda} \langle y_3, \dots, y_r \rangle ,$$

where $\Lambda = \Lambda_3 \times \Lambda_4 \times \dots \times \Lambda_r$, we find that y_2 has conjugates in P_r of the form

$$y_2^{y_3^{k_3} \dots y_r^{k_r}} = \prod_{j=1}^{p} (\ s+j,\ s+p+j,\ s+2p+j,\ \dots\ ,\ s+p^2-p+j\ )$$

where $s = k_r p^{r-1} + k_{r-1} p^{r-2} + \dots + k_3 p^2$,

and $0 \leq k_i \leq p-1$ for $i=3,\dots,r$.

These conjugates of y_2 have mutually disjoint supports, and there are precisely p^{r-2} of them. In fact, they generate the base group of $\langle y_2 \rangle \mathrm{wr}^{\Lambda} \langle y_3, \dots, y_r \rangle$, where $\Lambda = \Lambda_3 \times \Lambda_4 \times \dots \times \Lambda_r$.

Note that $\langle y_3, \dots, y_r \rangle \cong P_{r-2}$. The subgroup

$$\left\langle y_2^{y_3^{k_3} \dots y_r^{k_r}} \right\rangle$$

is the coordinate subgroup corresponding to $(k_3, k_4, \dots, k_r)$ of $\Lambda_3 \times \Lambda_4 \times \dots \times \Lambda_r$.

Ofcourse, y_2 has other conjugates of the form

$$y_2^{y_1^{k_1} y_3^{k_3} \cdots y_r^{k_r}}, \qquad \ldots\ldots\ldots(11)$$

but note

$$y_2^{y_1^{k_1} y_3^{k_3} \cdots y_r^{k_r}}$$

$$= (y_1^{-k_1} \; y_2 \; y_1^{k_1})^{y_3^{k_3} \cdots y_r^{k_r}},$$

$$= (y_1^{-k_1} \; (y_1^{y_2^{-1}})^{k_1} \; y_2)^{y_3^{k_3} \cdots y_r^{k_r}},$$

$$= (y_1^{y_3^{k_3} \cdots y_r^{k_r}})^{(-k_1)} (y_1^{y_2^{-1} y_3^{k_3} \cdots y_r^{k_r}})^{k_1} \; y_2^{y_3^{k_3} \cdots y_r^{k_r}},$$

i.e. a product of conjugates of y_1, and a conjugate of y_2 of the form described above in (11).

In general we obtain the following structure for P_r.

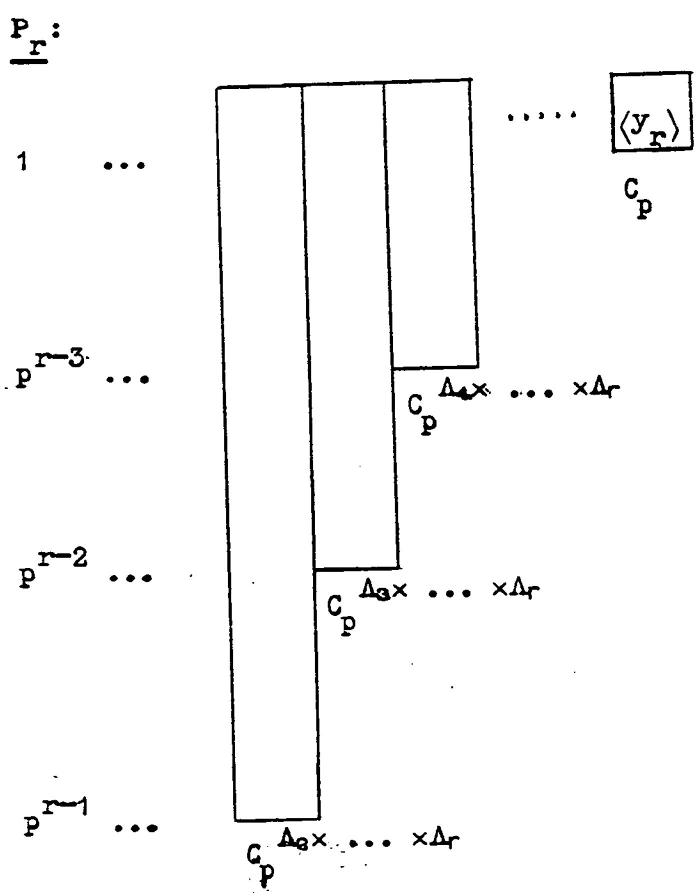


Fig. 2

The first column represents the base group of $P_r \left(\cong C_p \mathrm{wr}^{(p^{r-1})} P_{r-1}\right)$, and is of length p^{r-1} corresponding to the p^{r-1} conjugates of y_1, which generate this base group. The second column represents the base group of $\langle y_2, \ldots, y_r \rangle \cong C_p \mathrm{wr}^{(p^{r-2})} P_{r-2}$, which base group is generated by p^{r-2} conjugates of y_2, and so is of length p^{r-2}. In general the i th column is of length p^{r-i} and represents the base group of $\langle y_i, \ldots, y_r \rangle \cong C_p \mathrm{wr}^{(p^{r-i})} P_{r-i}$, and this base group is generated by p^{r-i} conjugates of y_i. Any element in P_r may be expressed uniquely in the form $f_1 f_2 \ldots f_r$ where f_i belongs to the base group $C_p^{\Delta_{i+1} \times \ldots \times \Delta_r}$ of $\langle y_i, \ldots, y_r \rangle$.

We will meet this table later in Chapter IV, both for P_r and its generalisation $C_{p^n} \mathrm{wr}^{(p^{r-1})} P_{r-1}$, and we will see that more of the structure of these groups can be fitted into the diagram. This will extend work done by Weir in [18] on P_r, for which the table is called a "partition diagram". The extra structure that can be represented in the diagram is the commutator structure of the group.

We denote the <u>commutator</u> $k_1^{-1} k_2^{-1} k_1 k_2$ of elements k_1, k_2 in the group G by $[k_1, k_2]$ as usual, and if K_1 and K_2 are two subgroups of G, we define

$$[K_1, K_2] = \langle [k_1, k_2] : k_1 \in K_1, k_2 \in K_2 \rangle .$$

Then the <u>lower central series</u> (L.C.S.) of the group G is given by $\gamma_1(G) = G$,

$$\gamma_i(G) = [\, \gamma_{i-1}(G),\ G\,] \quad \text{for } i \in \mathbb{Z}^+,\ i \geqslant 2 .$$

We call $\gamma_i(G)$ the <u>i-th term</u> in the series, and if for some i in $\mathbb{Z}^+$

$$\gamma_i(G) > \gamma_{i+1}(G) = \langle 1 \rangle$$

then G is said to be <u>nilpotent</u> of <u>(nilpotency) class</u> i , and we denote this class by $c(G)$.

If G is a group of order p^r then G is of <u>maximal class</u> if $c(G) = r-1$.

An element in $\gamma_i(G) \setminus \gamma_{i+1}(G)$ is said to have <u>(nilpotency) weight</u> i .

We will use left-normed notation, so that

$$[g_1, g_2, \ldots , g_n] = [\cdots[[\, g_1, g_2], g_3], \ldots , g_n] \qquad \ldots\ldots.(12)$$

and denote $[\, g_1, \underbrace{g_2, \ldots , g_2}_{n \text{ times}}]$ by $[\, g_1,\, _n\, g_2\,]$.

A commutator of type (12) is said to be a <u>simple commutator</u> in the <u>components</u> $g_1, \ldots, g_n$ of <u>length</u> n . For the definition of a <u>complex commutator</u> see below in the section on cpp-commutators, on p.26 .

In a nilpotent group a non-trivial commutator is said to be of <u>maximal length</u> if its length is equal to the nilpotency class of the group.

Recall the well-known identities, proved in [4] on p.43 :

1.1 LEMMA

Let x, y, z be elements of a group. Then

i) $[xy, z] = [x, z][x, z, y][y, z] = [x, z]^y[y, z]$;

ii) $[x, yz] = [x, z][x, y][x, y, z] = [x, z][x, y]^z$. $\square$

We denote the centre of the group G by

$$Z(G) = \{ g \in G : gg' = g'g \quad \forall g' \in G \} .$$

The upper central series (U.C.S.) of G is given by

$$Z_1(G) = Z(G) ,$$

$$Z_{i+1}(G)/ Z_i(G) = Z(G/ Z_i(G)) \quad \text{for } i \in \mathbb{Z}^+,$$

The group G is of nilpotency class $c(G)$ if and only if the upper central series of G terminates, and,

$$Z_{c(G)-1}(G) < Z_{c(G)}(G) = G .$$

In §1 of [17] , Shield defines a whole class of descending central series for an arbitrary group G :

Let a, b, and e be integers such that $a \geqslant b \geqslant 0$, $a \geqslant 1$ and $e \in \{1, p\}$ where p is a fixed prime. Define a weight relation $\rho(a, b, e)$ to be the least subset of $G \times \mathbb{N}$ satisfying the following conditions :

i) for all g in G , $(g, a) \in \rho(a, b, e)$;

ii) $(g, u) \in \rho(a, b, e) \Rightarrow (g^{-1}, u) \in \rho(a, b, e)$;

iii) $(g, u) \in \rho(a, b, e)$ and $(h, v) \in \rho(a, b, e) \Rightarrow$

$([g, h], u+v) \in \rho(a, b, e)$;

iv) a) $(g, u) \in \rho(a, b, 1) \Rightarrow (g^p, u+b) \in \rho(a, b, 1)$; and

b) $(g, u) \in \rho(a, b, p) \Rightarrow (g^p, pu) \in \rho(a, b, p)$;

v) $(g, u) \in \rho(a, b, e)$ and $(h, v) \in \rho(a, b, e) \Rightarrow$

$(gh, \min\{u,v\}) \in \rho(a, b, e)$.

Now define a function $w^e_{a,b} : G \longrightarrow \mathbb{Z}^+ \cup \{\infty\}$ by

$$w^e_{a,b}(g) = \begin{cases} \max\{ u : (g, u) \in \rho(a, b, e)\} & \text{if this maximum exists ,} \\ \infty & \text{if no such maximum exists .} \end{cases}$$

Then $w^e_{a,b}(g)$ is the <u>(a, b, e)-weight of g</u> and we define the <u>i-th term</u> , or the <u>i-th weight subgroup</u> of the (a, b, e)-series by

$$\gamma_i^{a,b,e}(G) = \{ g \in G : w^e_{a,b}(g) \geq i\} ,$$

which is easily checked to be a fully invariant subgroup.

Then the <u>(a, b, e)-series</u> of the group G is

$$G = \gamma_1^{a,b,e}(G) \geq \gamma_2^{a,b,e}(G) \geq \ldots \geq \gamma_i^{a,b,e}(G) \geq \ldots .$$

If there exists i in $\mathbb{Z}^+$ such that

$$\gamma_i^{a,b,e}(G) > \gamma_{i+1}^{a,b,e}(G) = \langle 1 \rangle$$

then we say G is <u>(a, b, e)-nilpotent</u> of <u>(a, b, e)-class</u> i .

The two most important cases are the (1,0,1)- and (1,0,p)- series. The (1,0,1)-series is just the lower central series, with $\gamma_i(G) = \gamma_i^{1,0,1}(G)$ for i in $\mathbb{Z}^+$. The (1,0,p)-series is the <u>cpp</u> (commutator-pth-power)-<u>series</u>, and we write for i in $\mathbb{Z}^+$

$$\pi_i(G) = \gamma_i^{1,0,p}(G) .$$

If the group G is cpp-nilpotent we denote the <u>cpp-class</u> of G by $d(G)$.

An important general result about (a, b, e)-series is

<u>1.2 LEMMA</u> (Corollary 1.3 [17])

The i-th term $\gamma_i^{a,b,e}(G)$ is generated modulo $\gamma_{i+1}^{a,b,e}(G)$ by elements of the form g^{p^t} where g is a commutator of nilpotency weight u in G, p is the fixed prime for the (a, b, e)-series, and for $e = 1$,

$$au + bt = i ,$$

or, for $e = p$,

$$aup^t = i . \qquad \square$$

We will be using the following immediate corollaries.

<u>1.3 COROLLARY</u>

i) If G is cpp-nilpotent then

$$d(G) = \max\{ wp^{s(w)-1} : 1 \leqslant w \leqslant c(G) \}$$

where a commutator of G of (nilpotency) weight w has order at most $p^{s(w)}$.

ii) The group G is cpp-nilpotent if and only if G is a nilpotent p-group of finite exponent . $\square$

<u>1.4 COROLLARY</u>

i) If G is (a, b, 1)-nilpotent then G has (a, b, 1)-class

$$\max\{ aw + b(s(w) - 1) : 1 \leqslant w \leqslant c(G) \}$$

where a commutator of (nilpotency) weight w has order at most $p^{s(w)}$, and p is the fixed prime for the (a, b, 1)-series.

ii) If $b \neq 0$ The group G is (a, b, 1)-nilpotent if and only if G is a nilpotent p-group of finite exponent for p the fixed prime of the (a, b, 1)-series. $\square$

We now look at the cpp-series in greater detail. From the definition we have immediately

$$[\ \pi_i(G),\ \pi_j(G)\ ] \leq \pi_{i+j}(G)$$

and

$$(\pi_i(G))^p = \langle g^p\ :\ g \in \pi_i(G) \rangle \leq \pi_{pi}(G)\ .$$

We will see shortly that for the cpp-series conditions i)-v) above can be recast in terms of cpp-commutators. We define cpp-commutators recursively, generalising the standard notion of a complex commutator as given by P.Hall on p.43 of [4] . Let $g_1, g_2, \dots, g_n$ be elements of the group G. We say that each g_i is a cpp-commutator of length 1 in the components $g_1, \dots, g_n$. Now suppose we have defined all cpp-commutators in the components $g_1, \dots, g_n$ of length less than ℓ. Then the cpp-commutators in the components $g_1, \dots, g_n$ of length ℓ are of two kinds. The first kind is of the form $[\underline{h}_1, \underline{h}_2]$ where $\underline{h}_1$ and $\underline{h}_2$ are cpp-commutators in the components $g_1, \dots, g_n$ of length ℓ_1 and ℓ_2 respectively, such that $\ell_1+\ell_2 = \ell$. The second kind occurs precisely when ℓ is divisible by p, and is then of the form $\underline{h}^p$ where $\underline{h}$ is a cpp-commutator in the components $g_1, \dots, g_n$ of length ℓ/p.

The standard definition of a complex commutator just excludes cpp-commutators of the second kind. Note that we have used the word "length" rather than "weight", which is used by P.Hall in the above reference, to avoid confusion between the "length" of a commutator and its weight as an element of the group.

We will call a cpp-commutator of the form

$$[[\cdots[\,g_1{}^{p^{t_1}},\ g_2]^{p^{t_2}},\ \ldots\ ,\ g_{n-1}]^{p^{t_{n-1}}},\ g_n]^{p^{t_n}}$$

a <u>simple cpp-commutator</u> . If $t_1=t_2=\ldots=t_n=0$, we clearly have just a simple commutator.

It is easy to see from the definition of the cpp-commutator that $(g, u) \in \rho(1,0,p)$ if and only if g can be expressed as a product of cpp-commutators in some components $g_1, \ldots, g_n$ where at least one of these cpp-commutators has length u and all others have length at least u . Thus $\pi_i(G)$ consists of those elements of G which can be expressed as the product of cpp-commutators, in some components $g_1, \ldots, g_n$, each of these cpp-commutators of length at least i , that is , for $i\geq 2$,

$$\pi_i(G) = \left\langle [\ \pi_{i-k}(G),\ \pi_k(G)\ ]\ ,\ (\pi_{\lceil \frac{i}{p} \rceil}(G))^p\ :\ k=1, \ldots, i-1 \right\rangle. \qquad \ldots\ldots.(13)$$

However, in certain cases this can be improved. For example, for finite p-groups Jennings [7] has shown that this series - his $\underline{\underline{K}}$-series - is equivalent to his $\mathcal{M}$-series defined by

$$\mathcal{M}_1(G) = G\ ,$$
$$\mathcal{M}_i(G) = \left\langle [\mathcal{M}_{i-1}(G),\ G]\ ,\ (\mathcal{M}_{\lceil \frac{i}{p} \rceil}(G))^p \right\rangle\ , \quad i \in \mathbb{Z}^+,\ \ i\geq 2\ .$$

Thus when G is a finite p-group,

$$\pi_i(G) = \left\langle [\pi_{i-1}(G),\ G]\ ,\ (\pi_{\lceil \frac{i}{p} \rceil}(G))^p \right\rangle \quad \text{for } i \in \mathbb{Z}^+,\ i\geq 2\ . \qquad \ldots\ldots.(14)$$

We will only be using the cpp-series of finite p-groups, but we give the generalisation of (14) to nilpotent p-groups of finite exponent, which by Corollary 1.3 are precisely the cpp-nilpotent groups. For this we require the following well-known lemma.

1.5 LEMMA (Lemma 1.7 [5])

Let the group G be generated by a set A. Then $\gamma_n(G)$ is generated by $\gamma_{n+1}(G)$ together with all commutators $[a_1, \dots , a_n]$ with each a_i in A. $\square$

1.6 LEMMA

Let G be a nilpotent group of exponent p^n. Then for i in $\mathbb{Z}^+$, $i \geq 2$,

$$\pi_i(G) = \left\langle [\ \pi_{i-1}(G), G\ ]\ , (\pi_{\lceil \frac{i}{p} \rceil}(G))^p \right\rangle . \qquad \ldots\ldots.(15)$$

Proof

We show that for $k = 1, \dots , i-1$,

$$[\pi_{i-k}(G), \pi_k(G)] \leq [\pi_{i-1}(G), G] .$$

Let g be an element of $[\pi_{i-k}(G), \pi_k(G)]$. Then g is a finite product of cpp-commutators of the form $[g_1, g_2]$ where $g_1 \in \pi_{i-k}(G)$ and $g_2 \in \pi_k(G)$. Now since G is cpp-nilpotent by Corollary 1.3, we have by Lemma 1.2 that each of g_1, g_2 can be expressed as a finite product of cpp-commutators of finite length, with each entry from G of cpp-weight 1. Let H be the subgroup of G generated by this finite set A of entries from G. Let A_i be the set of all commutators of the form $[a_1,\dots,a_i]$ with each a_j in A. Then A_i is a finite set. Note since G is of finite exponent p^n, each $[a_1, \dots , a_i]$ has order at most p^n. Then by Lemma 1.5, $\gamma_i(H)/\gamma_{i+1}(H)$ is finitely generated by the set $A_i\, \gamma_{i+1}(H)$, and since $\gamma_i(H)/\gamma_{i+1}(H)$ is abelian, it follows that any element of $\gamma_i(H)/\gamma_{i+1}(H)$ is of the form

$$h_1^{t_1} h_2^{t_2} \dots h_{|A|^i}^{t_{|A|^i}} \quad \gamma_{i+1}(H)$$

where $A_i = \{ h_j : j = 1, \dots, |A|^i \}$, and for $j=1,\dots,|A|^i$, $0 \leqslant t_j < p^n$. Thus $[\gamma_i(H):\gamma_{i+1}(H)]$ is finite. But

$$|H| = [H:\gamma_2(H)][\gamma_2(H):\gamma_3(H)] \dots [\gamma_{c(H)}(H): \langle 1 \rangle] ,$$

and so H is a finite p-group. Then by (14),

$$[g_1, g_2] \in [\pi_{i-1}(H), H] \leqslant [\pi_{i-1}(G), G]$$

which implies $g \in [\pi_{i-1}(G), G]$ as required. □

Obviously (14) and (15) make the construction of the cpp-series of cpp-nilpotent groups easier than if we had just (13).

<u>1.7 REMARK</u>

Note that the cpp-class is at least as great as the nilpotency class, and that the cpp-series and lower central series coincide for groups of exponent p. If G is a nilpotent group for which the lower and upper central series coincide, for example if G is of maximal class, then the cpp-series coincides with the lower central series : for, the cpp-series is a central series, and by a well-known result, for which see 10.2.2 of [3], if $G = Z^1(G) \geqslant Z^2(G) \geqslant \dots \geqslant Z^i(G) \geqslant \dots$
is a central series of G, then

$$\gamma_i(G) \leqslant Z^i(G) \leqslant Z_{c(G)-i+1}(G) \qquad \text{for } i=1,2,\dots,c(G) .$$

Note also that in general we may have $\pi_i(G) = \pi_{i+1}(G)$ <u>without</u> implying $\pi_j(G) = \pi_i(G)$ for all $j>i$. □

1.8 EXAMPLE

We give the cpp-series of an abelian group A of exponent p^n :

$$\pi_1(A) = A \ ,$$
$$\pi_2(A) = \pi_3(A) = \dots = \pi_p(A) = A^p = \langle g^p : g \in A \rangle \ ,$$
$$\vdots$$
$$\pi_{p^i+1}(A) = \pi_{p^i+2}(A) = \dots = \pi_{p^{i+1}}(A) = A^{p^{i+1}} \ ,$$
$$\vdots$$
$$\pi_{p^{n-1}+1}(A) = A^{p^n} = \langle 1 \rangle \ ,$$

so $d(A) = p^{n-1}$. □

Let G be a finite p-group, and let $|\pi_i(G)| = p^{n_i}$. Then we define Shield's constant (§ 2.1 [17])

$$a(G) = 1 + (p-1)\,\Sigma\,\{ n_i : 1 \leq i \leq d(G) \} \ .$$

1.9 EXAMPLE

For example,

$$\begin{aligned} a(C_{p^n}) &= 1 + (p-1)\,\Sigma\,\{ n_i : 1 \leq i \leq p^{n-1} \} \text{ from Example 1.8,} \\ &= 1 + (p-1)\{n + (p-1)(n-1) + p(p-1)(n-2) + p^2(p-1)(n-3) + \\ &\qquad \dots + p^{n-2}(p-1)\} \ , \\ &= 1 + (p-1)\{1 + p(1 + p(1 + p(\dots + p(1\underbrace{)) \cdots))}_{n\text{-}1\text{brackets}} \} \ , \\ &= 1 + (p-1)(1+p+p^2+ \dots +p^{n-1}) \ , \\ &= p^n \ . \end{aligned}$$

□

We show later in Corollary 3.12 that if G is a group of order p^n then $d(G) \leq p^{n-1}$ and $a(G) \leq p^n$ which by the above Examples 1.8 and 1.9 are best possible.

CHAPTER II : Nilpotency of the permutational wreath product, commutator constructions for C_{p^n} wr B , where B is a finite abelian p-group, and the centre of the permutational wreath product.

In 1959 Baumslag discovered ~~the~~ necessary and sufficient conditions on groups A and B for the standard wreath product A wr B to be nilpotent :

2.1 THEOREM (§ 3 [1])

Let A and B be non-trivial groups. Then A wr B is nilpotent if and only if A is a nilpotent p-group of finite exponent and B is a finite p-group. $\square$

This shows the unrestricted wreath product A Wr B is nilpotent if and only if A Wr B = A wr B , since A wr B is a subgroup of A Wr B, and B is a finite ~~p~~-group if and only if A Wr B = A wr B .

Shield has obtained an exact expression for the nilpotency class of the standard wreath product, with a very long and involved proof :

2.2 THEOREM (Corollary 5.5 [17])

Let B be a finite p-group, and A be a nilpotent p-group of class r such that for $1 \leqslant w \leqslant r$ the maximum order of a commutator of weight w in A is $p^{s(w)}$. Then A wr B is nilpotent with class precisely

$$c(A \text{ wr } B) = \max\{\, a(B)w + (p-1)d(B)(\, s(w)-1\,) : 1 \leqslant w \leqslant r \,\} .$$

$\square$

Note from Corollary 1.4 , $c(A \text{ wr } B) = (a(B), (p-1)d(B), 1)$-class of A , which provides a clue to the proof, to which we will return in Chapter VII .

Building on Baumslag's result and the following theorem of Meldrum, Scott established necessary and sufficient conditions on the group A and the pair "Λ , B" for $A\text{wr}^{\Lambda}B$ to be nilpotent.

2.3 THEOREM ([10])

Let A be a group and let "Λ , B" be a pair. Then $A\text{wr}^{\Lambda}B$ is nilpotent if and only if B is nilpotent and the groups $A\text{wr}^{\theta}\{B/\,St_B(\theta)\}$ are nilpotent of bounded class for each orbit θ of Λ .

2.4 THEOREM (Theorem 3.3.1 [14])

Let A be a group and let "Λ , B" be a pair. Then $W = A\text{wr}^{\Lambda}B$ is nilpotent if and only if one of three mutually exclusive situations exists :

i) A is trivial and B is nilpotent;

ii) A is a non-trivial nilpotent group, but not a p-group of finite exponent, B is nilpotent and $St_B(\Lambda) = B$;

iii) A is a non-trivial nilpotent p-group of finite exponent, B is a nilpotent group, and there exists n in $\mathbb{N}$ such that $|B/\,St_B(\theta)|$ divides p^n for all orbits θ of Λ .

Thus in case i) we have $c(A\text{wr}^{\Lambda}B) = c(B)$, and in case ii) $c(A\text{wr}^{\Lambda}B) = \max\{\, c(A),\, c(B)\,\}$. As we will show, we have in general, for cases i), ii), and iii), that

$$c(A\mathrm{wr}^{\Lambda}B) = \max\{\, c(B),\ c(A\mathrm{wr}^{\theta}\{B/\ St_B(\theta)\} \quad \text{where } \theta \text{ is an orbit of } \Lambda \,\} .$$

This part of the proof is essentially the theorem of Meldrum. We provide the proof given by Scott, which is similar to that given by Meldrum. We start with two lemmas of Scott [14] .

<u>2.5 LEMMA</u> (Lemma 3.1.4 [14])

$$\{A\mathrm{wr}^{\Lambda}B\}/\ St_B(\Lambda) \cong A\mathrm{wr}^{\Lambda}\{B/\ St_B(\Lambda)\} .$$

<u>Proof</u>

Note that for $f \in A^{\Lambda} \leq A\mathrm{wr}^{\Lambda}\{B/\ St_B(\Lambda)\}$, $f^{St_B(\Lambda)} = f$, with the usual action of $B/\ St_B(\Lambda)$ on Λ . Note also that in general

$$St_B(\Lambda) \lhd W = A\mathrm{wr}^{\Lambda}B ,$$

for if $f \in A^{\Lambda}$, $b \in B$, and $\beta \in St_B(\Lambda)$,

$$\begin{aligned}(fb)^{-1}\beta(fb) &= b^{-1}f^{-1}\beta fb ,\\ &= b^{-1}\beta f^{-\beta}fb ,\\ &= b^{-1}\beta b \in St_B(\Lambda) .\end{aligned}$$

We identify A^{Λ} in $A\mathrm{wr}^{\Lambda}B$ with A^{Λ} in $A\mathrm{wr}^{\Lambda}\{B/\ St_B(\Lambda)\}$ in the obvious way, and let $\theta : A\mathrm{wr}^{\Lambda}B \longrightarrow A\mathrm{wr}^{\Lambda}\{B/\ St_B(\Lambda)\}$ be given by

$$(fb)\theta = f(b\ St_B(\Lambda)) .$$

It is easy to see θ is well-defined, and is surjective. We also have that θ is a homomorphism, for if $f, g \in A^{\Lambda}$ and $b_1, b_2 \in B$ then

$$\begin{aligned}(fb_1gb_2)\theta &= (fg^{b_1^{-1}}\ b_1b_2)\theta ,\\ &= fg^{b_1^{-1}}\ (b_1b_2\ St_B(\Lambda)) ,\\ &= fg^{b_1^{-1}St_B(\Lambda)}\ (b_1\ St_B(\Lambda))(b_2\ St_B(\Lambda)) ,\end{aligned}$$

$$(fb_1gb_2)\theta = f(b_1 \ St_B(\Lambda))\ g(b_2 \ St_B(\Lambda)) ,$$
$$= (fb_1)\theta \ (gb_2)\theta .$$

The identity of $Awr^{\Lambda}\{B/ St_B(\Lambda)\}$ is $1.St_B(\Lambda)$, and

$$(fb)\theta = St_B(\Lambda) \iff f=1 \text{ and } b\in St_B(\Lambda) ,$$

i.e. $\ker\theta = St_B(\Lambda)$, and we have the result. $\square$

2.6 LEMMA (Lemma 3.3.9 [14])

Let A be a group and let "Λ , B" be a pair. Let $W = Awr^{\Lambda}B$, and let the faithful part of W be $W_f = Awr^{\Lambda}\{B/ St_B(\Lambda)\}$. Then

i) $\gamma_n(W) = [\ A^{\Lambda},\ _{n-1}W\ ]\ \gamma_n(B)$ for $n \in \mathbb{Z}^+$;

ii) $[\ A^{\Lambda},\ _nW\ ] \cong [\ A^{\Lambda},\ _nW_f\ ]$ for $n \in \mathbb{Z}^+$;

iii) Suppose A is the direct product of a family of groups $\{\ A_i\ :\ i \in I\ \}$. Then

$$[\ A^{\Lambda},\ _nW\ ] = \underset{i\in I}{\mathrm{Dr}}\ \mathrm{Dr}\{\ [\ A_i^{\theta},\ _nA_i^{\theta}B\ ]\ : \theta \text{ is an orbit of } \Lambda\}$$

for all $n \in \mathbb{N}$.

Proof

i) This part was also proved for split extensions in general by Weir as Theorem 1 of [18].

We proceed by induction on n .

n=1 : $\gamma_1(W) = W = A^{\Lambda}B = [\ A^{\Lambda},\ _0W\ ]\ \gamma_1(B)$.

We assume the result holds for some $n \geq 1$. Then

$$\gamma_{n+1}(W) = [\ \gamma_n(W),\ W\ ] ,$$
$$= [\ [\ A^{\Lambda},\ _{n-1}W\ ]\gamma_n(B),\ W\ ] \quad \text{by hypothesis,}$$
$$= \langle [\ A^{\Lambda},\ _{n-1}W,\ W\ ]^{\gamma_n(B)},\ [\ \gamma_n(B),\ W\ ]\rangle$$

by Lemma 1.1 i) ,

$$= [\ A^{\Lambda},\ _nW\ ][\ \gamma_n(B),\ A^{\Lambda}B\ ] \quad \text{since } [\ A^{\Lambda},\ _nW\ ] \lhd W ,$$

$$\gamma_{n+1}(W) = [\, A^{\Lambda},\ {}_nW\,] \left\langle [\, \gamma_n(B),\ B\,]\ ,\ [\, \gamma_n(B),\ A^{\Lambda}\,]^B \right\rangle$$

by Lemma 1.1 ii) ,

$$= [\, A^{\Lambda},\ {}_nW\,][\, A^{\Lambda},\ \gamma_n(B)\,]\gamma_{n+1}(B)$$ since B normalises A^{Λ} and $\gamma_n(B)$,

$$= [\, A^{\Lambda},\ {}_nW\,]\ \gamma_{n+1}(B)$$ since $[A^{\Lambda},\ \gamma_n(B)] \leqslant [A^{\Lambda},\ {}_nW]$, which is a corollary to Theorem 10.3.6 of [3] .

ii) This follows directly from the proof of Lemma 2.5.

iii) Recall that $A_i\mathrm{wr}^{\theta}B = A_i{}^{\theta}B \leqslant W$ for $i\in I$, θ an orbit of Λ ,

since $A_i{}^{\theta} = \left\langle A_{i,\lambda} : \lambda \in \theta \right\rangle$.

We first prove by induction on n that

$$[\, A^{\Lambda},\ {}_nW\,] = \underset{i\in I}{\mathrm{Dr}}\ \mathrm{Dr}\{\ [\, A_i{}^{\theta},\ {}_nW\,] : \theta \text{ is an orbit of } \Lambda\,\}$$

for n in $\mathbb{N}$.

<u>$n=0$</u> : $[\, A^{\Lambda},\ {}_oW\,] = A^{\Lambda} = \underset{i\in I}{\mathrm{Dr}}\ \mathrm{Dr}\{\ A_i{}^{\theta} : \theta \text{ is an orbit of } \Lambda\,\}$

$$= \underset{i}{\mathrm{Dr}}\ \underset{\theta}{\mathrm{Dr}}\ [\, A_i{}^{\theta},\ {}_oW\,]\ .$$

We assume the result holds for some $n \geqslant 0$. Then by hypothesis

$$[\, A^{\Lambda},\ {}_{n+1}W\,] = [\ \underset{i\in I}{\mathrm{Dr}}\ \underset{\theta \text{ an orbit}}{\mathrm{Dr}}\ [\, A_i{}^{\theta},\ {}_nW\,],\ W\,]\ .$$

Now $A_i{}^{\theta} \lhd W$, so $[\, A_i{}^{\theta},\ {}_nW\,] \leqslant A_i{}^{\theta}$ for $i\in I$, θ an orbit of Λ.

Recall $[\, A_i,\ A_j\,] = \langle 1 \rangle$ for $i\neq j$ and $[\, A^{\Sigma},\ A^{\Psi}\,] = \langle 1 \rangle$ if Σ and Ψ are distinct orbits of Λ . Hence by Lemma 1.1 i) ,

$$[\ \underset{i\in I}{\mathrm{Dr}}\ \underset{\theta \text{ an orbit}}{\mathrm{Dr}}\ [\, A_i{}^{\theta},\ {}_nW\,],\ W\,]$$

$$= \left\langle [\, A_i{}^{\theta},\ {}_nW,\ W\,] : i\in I\ ,\ \theta \text{ an orbit of } \Lambda \right\rangle\ ,$$

$$= \underset{i}{\mathrm{Dr}}\ \underset{\theta}{\mathrm{Dr}}\ [\, A_i{}^{\theta},\ {}_{n+1}W\,]\ ,$$

as required for the induction.

We again use $[A_i, A_j] = [A^{\Sigma}, A^{\Psi}] = \langle 1 \rangle$ for $i \neq j$ and Σ, Ψ distinct orbits of Λ, to obtain from Lemma 1.1 ii),

$$\mathop{\mathrm{Dr}}_{i \in I} \ \mathop{\mathrm{Dr}}_{\theta \text{ an orbit}} [A_i^{\theta}, {}_nW] = \mathop{\mathrm{Dr}}_{i} \ \mathop{\mathrm{Dr}}_{\theta} [A_i^{\theta}, {}_nA_i^{\theta}B]$$

for all n in $\mathbb{N}$, as required. □

2.7 COROLLARY

$$c(A\mathrm{wr}^{\Lambda}B) = \max\{ c(B) , c(A\mathrm{wr}^{\Lambda}\{B/ \mathrm{St}_B(\Lambda)\}) \} .$$

Proof

Let $W_f = A\mathrm{wr}^{\Lambda}\{B/ \mathrm{St}_B(\Lambda)\}$. By Lemma 2.6 i),

$$\gamma_n(W_f) = [A^{\Lambda}, {}_{n-1}W_f] \, \gamma_n\{B/ \mathrm{St}_B(\Lambda)\} .$$

Note $c(B) \geq c(B/ \mathrm{St}_B(\Lambda))$. Then by Lemma 2.6 i), ii), we obtain the result. □

Proof of Theorem 2.3

The necessary condition is immediate since if $W = A\mathrm{wr}^{\Lambda}B$ is nilpotent, then its subgroups must be nilpotent, and from Corollary 2.7,

$$c(A\mathrm{wr}^{\theta}B) = \max\{ c(B) , c(A\mathrm{wr}^{\theta}\{B/ \mathrm{St}_B(\theta)\}) \} \leq c(W) ,$$

for each orbit θ of Λ.

The sufficient condition also follows easily. Let B be a nilpotent group, and suppose $W_{\theta f} = A\mathrm{wr}^{\theta}\{B/ \mathrm{St}_B(\theta)\}$ is nilpotent for all orbits θ of Λ. Let $n = \max\{c(W_{\theta f}) : \theta \text{ an orbit of } \Lambda\}$, which exists by hypothesis. Let $W_{\theta} = A\mathrm{wr}^{\theta}B$, and let $m = \max\{ n , c(B) \}$. Then

$$\gamma_{m+1}(W) = [A^{\Lambda}, {}_mW] \, \gamma_{m+1}(B) \quad \text{by Lemma 2.6 i)} ,$$

$$\gamma_{m+1}(W) = \mathrm{Dr}\{[\, A^{\theta},\ {}_mW_\theta\,] : \theta \text{ is an orbit of } \Lambda\} \qquad \text{by Lemma 2.6 iii)},$$

$$\cong \mathop{\mathrm{Dr}}_{\theta} [\, A^{\theta},\ {}_mW_{\theta f}\,] \qquad \text{by Lemma 2.6 ii)},$$

$$= \langle 1 \rangle .$$

Hence W is nilpotent. □

2.8 REMARK

We observe that in the above proof, $\gamma_m(W) \neq \langle 1 \rangle$ and so W has class m , i.e. if $A\mathrm{wr}^{\Lambda}B$ is nilpotent,

$$c(A\mathrm{wr}^{\Lambda}B) = \max\{\, c(B)\ ,\ c(A\mathrm{wr}^{\theta}\{B/\ \mathrm{St}_B(\theta)\}) : \theta \text{ an orbit of } \Lambda \,\}.$$

□

For the proof of Theorem 2.4 we require several more results. The first three results are needed for the necessary condition of Theorem 2.4, in that we require $B/\ \mathrm{St}_B(\theta)$ to be of bounded order. The proofs of results from [14] are essentially those of Scott.

2.9 LEMMA (Lemma 3.3.6 [14])

Let A be a group, and let "Λ , B" be a pair. Let $W = A\mathrm{wr}^{\Lambda}B$, $f \in A^{\Lambda}$ and $b_1, \dots , b_k \in B$. Then

$$\sigma([f,\ b_1,\ \dots\ ,\ b_k]) \subseteq \sigma(f)B$$

and $|\sigma([f,\ b_1,\ \dots\ ,\ b_k])| \leq 2^k|\sigma(f)|$.

Proof

We proceed by induction on k .

k=1 : $[f,\ b_1] = f^{-1}f^{b_1}$, so $\sigma([f,\ b_1]) \subseteq \sigma(f^{-1}) \cup \sigma(f^{b_1})$,

$$= \sigma(f) \cup \sigma(f)b_1 ,$$

$$\subseteq \sigma(f)B ,$$

and $|\sigma([f, b_1])| \leqslant 2|\sigma(f)|$, as required.

Now suppose the result holds for some $k \geqslant 1$. Then

$$[f, b_1, \ldots , b_{k+1}] = [f, b_1, \ldots , b_k]^{-1}[f, b_1, \ldots , b_k]^{b_{k+1}};$$

so

$$\begin{aligned} &\sigma([f, b_1, \ldots , b_{k+1}]) \\ &\subseteq \sigma([f, b_1, \ldots , b_k]) \cup \sigma([f, b_1, \ldots , b_k])b_{k+1} , \\ &\subseteq \sigma(f)B \quad \text{by hypothesis,} \end{aligned}$$

and

$$\begin{aligned} |\sigma([f, b_1, \ldots , b_{k+1}])| &\leqslant 2|\sigma([f, b_1, \ldots , b_k])| , \\ &\leqslant 2^{k+1}|\sigma(f)| \qquad \text{by hypothesis.} \end{aligned}$$

Hence we have the result. □

2.10 LEMMA (Lemma 3.3.7 [14])

Let A be a non-trivial group, let "Λ , B" be a pair, and let $W = A\mathrm{wr}^{\Lambda}B$. Let Θ be an orbit of Λ , and suppose there exists n in $\mathbb{N}$ such that $2^n < |\Theta|$. Let $\theta \in \Theta$ and let $f \in A_\theta \setminus \langle 1 \rangle$. Then there exist elements $b_1, \ldots , b_{n+1}$ in B such that $[f, b_1, \ldots , b_{n+1}] \neq 1$.

Proof

We proceed by induction on n .

$\underline{n = 0}$: $1 < |\Theta|$, so there exists $\theta_1 \in \Theta$ such that $\theta_1 \neq \theta$. Θ is an orbit, so B is transitive on Θ , and thus there exists b_1 in B such that $\theta b_1 = \theta_1$. Then

$$[f, b_1](\theta_1) = f(\theta_1)^{-1}f(\theta_1)^{b_1} = f(\theta) \neq 1 .$$

Hence $\theta_1 \in \sigma([f, b_1])$, so $[f, b_1] \neq 1$, as required.

Now suppose the result holds for some $n \geq 0$. Then if $2^{n+1} < |\theta|$, we have $2^n < |\theta|$ and so by hypothesis there exist elements $b_1, \ldots, b_{n+1}$ in B such that $[f, b_1, \ldots, b_{n+1}] \neq 1$. Hence there exists $\mu \in \sigma([f, b_1, \ldots, b_{n+1}]) \subseteq \sigma(f)B \subseteq \theta$ by Lemma 2.9. Lemma 2.9 also gives us

$$|\sigma([f, b_1, \ldots, b_{n+1}])| \leq 2^{n+1}|\sigma(f)| = 2^{n+1} < |\theta|$$

by hypothesis, and hence there exists $\lambda \in \theta \setminus \sigma([f, b_1, \ldots, b_{n+1}])$. The group B is transitive on θ, so there exists b_{n+2} in B such that $\mu b_{n+2} = \lambda$. Hence

$$\begin{aligned} &[f, b_1, \ldots, b_{n+1}, b_{n+2}](\lambda) \\ &= [f, b_1, \ldots, b_{n+1}]^{-1}(\lambda)\, [f, b_1, \ldots, b_{n+1}](\lambda b_{n+2}^{-1}), \\ &= [f, b_1, \ldots, b_{n+1}](\mu) \neq 1 . \end{aligned}$$

Hence $[f, b_1, \ldots, b_{n+2}] \neq 1$ and we have the result by induction. $\square$

<u>2.11 COROLLARY</u> (Corollary 3.3.8 [14])

Let A be a non-trivial group, let "Λ , B" be a pair, and let $W = A\mathrm{wr}^{\Lambda}B$ be a nilpotent group of nilpotency class $c(W)$. Then each orbit θ of Λ is finite, and hence $B/\,\mathrm{St}_B(\theta)$ is finite for all orbits θ, with

$$|B/\,\mathrm{St}_B(\theta)| \leq 2^{c(W)}!$$

for all orbits θ of Λ.

<u>Proof</u>

Let θ be an orbit of Λ, and suppose $2^{c(W)} < |\theta|$. Then by Lemma 2.10, $\gamma_{c(W)+2}(W) \neq \langle 1 \rangle$, which is a contradiction. Thus

$$|\theta| \leq 2^{c(W)} \qquad \text{for all orbits } \theta \text{ of } \Lambda .$$

Hence since $B/\,St_B(\theta)$ is a permutation group on θ , i.e. a subgroup of S_θ ,

$$|B/\,St_B(\theta)| \leq 2^{c(W)}\,! \quad \text{for all orbits } \theta \text{ of } \Lambda . \qquad \square$$

The following five results are required to show that the conditions of iii) in Theorem 2.4 are sufficient for $W = A\,wr^{\Lambda} B$ to be nilpotent. The first result is well-known, and a proof may be found in § 7.2 of [8] .

2.12 LEMMA

Let G be a group. Then every subgroup H of G induces a transitive permutational representation of G by assigning to any element g' of G the permutation $Hg \longrightarrow Hgg'$ of the right cosets of H . All transitive permutational representations of G can be obtained in this way. $\square$

In other words, for every transitive pair "Λ , G" we can find a subgroup H_Λ of G such that we may regard Λ as the set of right cosets of H_Λ in G , upon which G acts by right multiplication .

2.13 REMARK

It is clear from the lemma that if B is a finite p-group and "Λ , B" is a transitive pair then $|\Lambda| = p^r$ for some r . $\square$

2.14 LEMMA (Lemma 3.3.12 [14])

Let A be a group. Let "Λ_1 , B" and "Λ_2 , B" be transitive pairs, i.e. representations of B, induced by subgroups H_1 and H_2 respectively, where $H_1 \leq H_2 \leq B$. Let $W_1 = A \operatorname{Wr}^{\Lambda_1} B$, and let $W_2 = A \operatorname{Wr}^{\Lambda_2} B$. Then there exists a monomorphism from W_2 into W_1.

Proof

We have from Lemma 2.12 that Λ_1 and Λ_2 may be regarded as the set of right cosets in B of H_1 and H_2 respectively. Let R be a right transversal to H_2 in B, and let S be a right transversal to H_1 in H_2. Then $SR = \{ sr : s \in S, r \in R \}$ is a right transversal to H_1 in B, and

$$\Lambda_1 = \{ H_1 sr : s \in S, r \in R \},$$

$$\Lambda_2 = \{ H_2 r : r \in R \},$$

with $(H_1 sr)b = H_1 srb$ and $(H_2 r)b = H_2 rb$ for all b in B.

We define $\theta : W_2 \longrightarrow W_1$ by $(fb)\theta = (f\theta)(b\theta)$ where

$$b\theta = b \qquad \forall\, b \in B,$$

and $(f\theta)(H_1 sr) = f(H_2 r)$ $\quad \forall\, s \in S, \forall\, r \in R$, and $\forall\, f \in \operatorname{Cr} A^{\Lambda_2}$.

(Note that if $[H_2 : H_1]$ is infinite then the support of $f\theta$ may be infinite even if the support of f is finite.)

The map $f\theta$ is well-defined. For if $s, s_1 \in S$, and $r, r_1 \in R$ are such that $H_1 sr = H_1 s_1 r_1$ then

$$srr_1^{-1}s_1^{-1} \in H_1 \leq H_2$$

$$\Rightarrow rr_1^{-1} \in H_2,$$

$$\Rightarrow r = r_1 \text{ since } R \text{ is a transversal to } H_2 \text{ in } B.$$

Thus $H_1 s = H_1 s_1$, which implies $s = s_1$ since S is a transversal to H_1 in H_2. Hence $f\theta$ is well-defined.

We now show θ is a homomorphism, i.e. for f,g in $\mathrm{Cr}\, A^{\Lambda_2}$ and b,c in B, $(fbgc)\theta = (fb)\theta\,(gc)\theta$. Now

$$(fbgc)\theta = (fg^{b^{-1}}bc)\theta = (fg^{b^{-1}})\theta\,(bc)\theta = (fg^{b^{-1}})\theta\,(b\theta)(c\theta)\ ,$$

so we need to show

$$(fg^{b^{-1}})\theta = (f\theta)\,(g\theta)^{(b\theta)^{-1}}\ .$$

First, $(g\theta)^{(b\theta)^{-1}} = (g^{b^{-1}})\theta$. For, let $s \in S$ and $r \in R$, and let $s_1 \in S$ and $r_1 \in R$ be such that $H_1srb = H_1s_1r_1$. Then

$$srbr_1^{-1}s_1^{-1} \in H_1 \leq H_2$$

$$\Rightarrow rbr_1^{-1} \in H_2\ ,$$

$$\Rightarrow H_2rb = H_2r_1\ .$$

Hence
$$\begin{aligned}(g\theta)^{(b\theta)^{-1}}(H_1sr) = (g\theta)^{b^{-1}}(H_1sr) &= (g\theta)(H_1srb)\ ,\\ &= (g\theta)(H_1s_1r_1)\ ,\\ &= g(H_2r_1)\ ,\\ &= g(H_2rb)\ ,\\ &= g^{b^{-1}}(H_2r)\ ,\\ &= (g^{b^{-1}})\theta(H_1sr)\ ,\end{aligned}$$

and so $(g\theta)^{(b\theta)^{-1}} = (g^{b^{-1}})\theta$.

Then
$$\begin{aligned}(fg^{b^{-1}})\theta(H_1sr) = (fg^{b^{-1}})(H_2r) &= f(H_2r)\,g^{b^{-1}}(H_2r)\ ,\\ &= (f\theta)(H_1sr)\,(g\theta)^{b^{-1}}(H_1sr)\ ,\\ &= (f\theta\,(g\theta)^{(b\theta)^{-1}})(H_1sr)\ ,\end{aligned}$$

which completes the proof that θ is a homomorphism.

We show the kernel of θ is trivial. Let $f \in \mathrm{Cr}\, A^{\Lambda_2}$ and $b \in B$, and suppose $(fb)\theta = 1$. Then $b = 1$ and so

$$f\theta = 1$$

$$\iff (f\theta)(H_1 sr) = 1 \qquad \forall\, s \in S,\ \forall\, r \in R\,,$$

$$\iff f(H_2 r) = 1 \qquad \forall\, r \in R\,,$$

$$\iff f = 1\,,$$

i.e. $\ker \theta = \langle 1 \rangle$. Hence θ is a monomorphism. □

<u>2.15 COROLLARY</u> (Corollary 3.3.13 [14])

Let A be a group, let "Λ , B" be a transitive pair, and let $W = A \text{ Wr}^{\Lambda} B$. Then we may embed W in $A \text{ Wr } B$.

<u>Proof</u>

Let "Λ , B" be the representation induced by the subgroup H of B . Now $A \text{ Wr } B = A \text{ Wr}^{\{B\}} B$ where "B , B" is the right regular representation induced by $\langle 1 \rangle \leq H$. Hence by Lemma 2.14 we may embed $A \text{ Wr}^{\Lambda} B$ in $A \text{ Wr } B$. □

Scott quotes the following theorem of Scruton :

<u>2.16 THEOREM</u> (Theorem 4.7 [15])

Let p be a prime, and let A be a nilpotent group of exponent p^k . Let B be a group of order p^t . Let $W = A \text{ wr } B$. Then

$$c(W) \leq \begin{cases} k\, c(A)\, p^{t-2}(2p-1) & \text{if } B \text{ is not cyclic} \\ c(A)\, p^{t-1}(kp-k+1) & \text{if } B \text{ is cyclic .} \end{cases}$$ □

Incidentally, this proves the sufficient condition for Theorem 2.1.

Shield's result Theorem 2.2 gives us

$$c(W) = \max\{a(B)w + (p-1)d(B)(\ s(w)-1\ ) : 1 \leq w \leq c(A)\ \}$$

where the order of a commutator in A of weight w is at most $p^{s(w)}$. Corollary 3.12 below gives us bounds for $a(B)$ and $d(B)$ in terms of the order of B, which gives us an alternative bound to that of Theorem 2.16, which could also be used in the following proposition:

2.17 PROPOSITION (Proposition 3.3.15 [14])

Let p be a prime, and let A be a nilpotent p-group of finite exponent. Let "Λ , B" be a pair such that $B/\,St_B(\theta)$ is a finite p-group for each orbit θ of Λ , of order bounded above by p^t for some t in $\mathbb{N}$ for all orbits θ of Λ . Let $W_{\theta f} = Awr^{\theta}\{B/\,St_B(\theta)\}$. Then $W_{\theta f}$ is nilpotent for all orbits θ , and there exists n in $\mathbb{Z}^+$ such that $c(W_{\theta f}) \leq n$ for all orbits θ .

Proof

Let θ be an orbit of Λ . The conditions on A and $B/\,St_B(\theta)$ imply that $Awr\{B/\,St_B(\theta)\}$ is nilpotent, by Theorem 2.1. Since $B/\,St_B(\theta)$ is finite, we also have $A\,Wr\{B/\,St_B(\theta)\}$ equals $Awr\{B/\,St_B(\theta)\}$, and $A\,Wr^{\theta}\{B/\,St_B(\theta)\}$ equals $Awr^{\theta}\{B/\,St_B(\theta)\}$. Hence by Corollary 2.15, $W_{\theta f}$ is nilpotent.

Since $|B/\,St_B(\theta)| \leq p^t$, we have by Theorem 2.16 that if the exponent of A is p^k then

$$c(Awr\{B/\,St_B(\theta)\}) \leq \begin{cases} k\,c(A)\,p^{t-2}(2p-1) & \text{if } B \text{ is not cyclic} \\ c(A)\,p^{t-1}(kp-k+1) & \text{if } B \text{ is cyclic} \end{cases}$$

$$= n\ , \quad \text{say.}$$

Hence by Corollary 2.15, $c(W_{\theta f}) \leq n$.

The orbit θ of Λ was arbitrarily chosen, and so we have the result. $\square$

In Theorem 2.4, Scott uses results she obtained about Baer groups to prove the necessary part for $W = A\mathrm{wr}^{\Lambda}B$ to be nilpotent ; a Baer group is a group which is generated by its abelian subnormal subgroups, for example, nilpotent groups. A more direct proof will be given here, for which we require the following lemma and a result about the centraliser of B in W.

2.18 LEMMA

Let A be a non-trivial group, and let "Λ , B" be a non-trivial pair. Then if $W = A\mathrm{wr}^{\Lambda}B$ is nilpotent, A is of finite exponent.

Proof

Suppose W is nilpotent. Then by Corollary 2.11, all orbits of Λ are finite. Since "Λ , B" is non-trivial, there exists λ in Λ and b in B such that $\lambda b \neq \lambda$. Let i be the least element in $\mathbb{Z}^+$ such that $\lambda b^i = \lambda$, and let $\Omega = \{\lambda b^j : 0 \leq j < i\}$, the orbit of Λ under $\langle b \rangle$ which contains λ . Let $W_b = A\mathrm{wr}^{\Lambda}\langle b \rangle$ with the action of $\langle b \rangle$ on Λ inherited from "Λ , B" . The group W_b embeds naturally in $A\mathrm{wr}^{\Lambda}B$, and so $c(W) \geq c(W_b)$. By Lemma 2.6 iii) ,

$$[A^{\Lambda}, {}_nW_b] = \mathrm{Dr}\ \{\ [A^{\theta}, {}_nA^{\theta}\langle b \rangle\ ]\ :\ \theta \text{ is an orbit of } "\Lambda\ ,\ \langle b \rangle"\}.$$

Hence,

$$c(W) \geq c(W_b) \geq c(A\mathrm{wr}^{\Omega}\langle b \rangle)\ ,$$

$$= \max\{\ c(\langle b \rangle)\ ,\ c(A\mathrm{wr}^{\Omega}\{\langle b \rangle / \mathrm{St}_{\langle b \rangle}(\Omega)\})$$

by Corollary 2.7 ,

$$c(W) \geq c(A\mathrm{wr}^{\Omega}\{\langle b\rangle / \mathrm{St}_{\langle b\rangle}(\Omega)\}) ,$$
$$= c(A\mathrm{wr}\langle b'\rangle) \quad \text{where} \quad |b'| = i .$$

But by Theorem 2.1 , $A\mathrm{wr}\langle b'\rangle$ is nilpotent only if A is of finite exponent . The result now follows . □

The following lemma generalises the result mentioned for the standard wreath product in § 3 of [13] .

2.19 LEMMA

Let A be a non-trivial group, and let "Λ , B" be a non-trivial pair. Let $\underline{W} = A\,\mathrm{Wr}^{\Lambda}B$ and let $W = A\mathrm{wr}^{\Lambda}B$. Then the subgroup of the base group of $\underline{W}$ (W) which centralises $B \setminus \mathrm{St}_B(\Lambda)$ is $\mathrm{Cr}\,\{\mathfrak{D}_{\mathrm{Cr}\,A^{\theta}} : \theta \text{ an orbit of } \Lambda\}$ ($\mathop{\mathrm{Dr}}_{\theta}\mathfrak{D}_{\mathrm{Dr}\,A^{\theta}}$), and hence this subgroup centralises B .

Proof

Let $f \in C_W(B) \cap \mathrm{Dr}\,A^{\Lambda}$. Then

$$[f, b] = 1 \qquad \forall\, b \in B\setminus \mathrm{St}_B(\Lambda)$$
$$\Longleftrightarrow f = f^{b} \qquad \forall\, b \in B\setminus \mathrm{St}_B(\Lambda) ,$$
$$\Longleftrightarrow f \in \mathrm{Dr}\,\{\mathfrak{D}_{\mathrm{Dr}\,A^{\theta}} : \theta \text{ is an orbit of } \Lambda\} .$$

If $f \in \mathrm{Dr}\,A^{\Lambda}$ and $b \in \mathrm{St}_B(\Lambda)$ then $[f, b] = f^{-1} f^{b} = 1$ and so $\mathop{\mathrm{Dr}}_{\theta}\mathfrak{D}_{\mathrm{Dr}\,A^{\theta}}$ centralises $\mathrm{St}_B(\Lambda)$, and we obtain the result for W .

The result for $\underline{W}$ is obtained in a similar fashion. □

Clearly if "Λ , B" is a transitive pair then

$$\mathfrak{D}_{\mathrm{Cr}\,A^{\Lambda}} = C_{\underline{W}}(B) \cap \mathrm{Cr}\,A^{\Lambda} , \text{ and}$$

$$\mathcal{D}_{\mathrm{Dr}\ A^\Lambda} = C_W(B) \cap \mathrm{Dr}\ A^\Lambda .$$

Proof of Theorem 2.4

=> : W is nilpotent. Hence by Theorem 2.3, B is nilpotent and the groups $A\mathrm{wr}^\theta\{B/\ \mathrm{St}_B(\theta)\}$ are nilpotent of bounded class for each orbit θ of Λ. The group A is nilpotent since A^Λ is a subgroup of W. Suppose now that W is not of type i) or type ii). We want to show that W is then of type iii).

If "Λ , B" is trivial, since by hypothesis W is not of type i) or type ii), A is a non-trivial nilpotent p-group of finite exponent.

Now suppose "Λ , B" is non-trivial. By hypothesis A is non-trivial, and so by Lemma 2.18 is of finite exponent. By Corollary 2.11, $B/\ \mathrm{St}_B(\theta)$ is finite for each orbit θ of Λ. Suppose for contradiction that A contains an element f of order q, and $B/\ \mathrm{St}_B(\theta)$ contains an element b of order p, where p and q are distinct primes not equal to 1. Recall that in a nilpotent group, periodic elements of coprime order commute, for which see the corollary to Theorem 1.10 of [5]. Choose $\mu\in\theta$ and identify A with A_μ. Then $[f, b] = 1$, and so since $B/\ \mathrm{St}_B(\theta)$ is faithful, by Lemma 2.19, f belongs to $\mathcal{D}_{A\theta}$. But f is in A_μ. Hence $f=1$, which is a contradiction. Thus A and $B/\ \mathrm{St}_B(\theta)$ are p-groups for the same prime p, and all orbits θ of Λ. Finally, there exists n in $\mathbb{N}$ such that $p^n \geq 2^{c(W)}!$, and so by Corollary 2.11 we have that $|B/\ \mathrm{St}_B(\theta)|$ divides p^n for all orbits θ of Λ . Thus we obtain the conditions of iii) , and so have proved the necessary part.

<u><=</u> : We give Scott's proof. Parts i) and ii) are clear. For part iii), by Proposition 2.17, $A\mathrm{wr}^{\theta}\{B/\ St_B(\theta)\}$ is nilpotent of bounded class for all orbits θ of Λ , and since B is nilpotent, we have by Theorem 2.3 that W is nilpotent . □

Note that Lemmas 2.5,2.6 and so Theorem 2.3 generalise immediately to $W = A\ \mathrm{Wr}^{\Lambda}B$, the only change required being the replacement of $A^{\Lambda} = \mathrm{Dr}\ A^{\Lambda}$ by $\mathrm{Cr}\ A^{\Lambda}$. Similarly, Lemma 2.18 generalises to the unrestricted wreath product. We then have

<u>2.20 COROLLARY</u>

Let A be a group and let "Λ , B" be a pair. Then $A\ \mathrm{Wr}^{\Lambda}B$ is nilpotent if and only if $A\mathrm{wr}^{\Lambda}B$ is nilpotent, and in the case of nilpotency,

$$c(A\ \mathrm{Wr}^{\Lambda}B) = c(A\mathrm{wr}^{\Lambda}B) .$$

<u>Proof</u>

<u>=></u> : $A\mathrm{wr}^{\Lambda}B \leqslant A\ \mathrm{Wr}^{\Lambda}B$.

<u><=</u> : By the generalisations of Lemma 2.6 iii) and Corollary 2.7 ,

$$c(A\ \mathrm{Wr}^{\Lambda}B) = \max\{\ c(B)\ ,\ c(A\ \mathrm{Wr}^{\theta}\{B/\ St_B(\theta)\})\ :\ \theta \text{ is an orbit of } \Lambda\ \} ,$$

and since $A\mathrm{wr}^{\Lambda}B$ by hypothesis is nilpotent, each orbit is finite, and so $A\ \mathrm{Wr}^{\theta}\{B/\ St_B(\theta)\} = A\mathrm{wr}^{\theta}\{B/\ St_B(\theta)\}$. Hence we have the result. □

2.21 REMARK

We have seen that Theorem 2.4 reduces finding the nilpotency class of the permutational wreath product of two pairs "Λ_1 , H_1" and "Λ_2 , H_2" to finding the nilpotency class of $A\mathrm{wr}^{\Lambda}B$, where A is a nilpotent p-group of finite exponent, B is a finite p-group, and "Λ , B" is a faithful transitive pair, i.e. B is a transitive subgroup of the Sylow p-subgroup of S_{Λ} . We have already seen in Remark 2.13 that this implies $\Lambda = (p^r)$ for some r , and so we are looking at transitive subgroups of P_r for some r . Using Theorem 2.2 on $P_r = \underbrace{C_p \mathrm{wr}\, C_p \mathrm{wr} \ldots \mathrm{wr}\, C_p}_{r\ C_p\text{'s}}$,

it is easy to prove by induction that the nilpotency class of P_r is p^{r-1} : recall from Example 1.9 that $a(C_p) = p$. This is an old result - see [18] - and is a special case of Theorem 3.3 below. □

We remark that Shield's result, Theorem 2.2 , generalises results obtained by Liebeck [9] . Liebeck's work will also be of great importance in determining the nilpotency class of $C_{p^n}\mathrm{wr}^{(p^2)}B$, where "(p^2) , B" is a faithful transitive pair. We now give these results.

2.22 THEOREM (Theorem 5.1 [9])

Let A be an abelian group of exponent p^n and let $B = \langle b_1 \rangle \times \langle b_2 \rangle \times \ldots \times \langle b_m \rangle$ be a direct product of m cyclic groups of orders $p^{\beta_1}, p^{\beta_2}, \ldots , p^{\beta_m}$ respectively, where $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_m$. Then $W = A \mathrm{wr}\, B$ has nilpotency class given by

$$c(W) = \sum_{i=1}^{m} (p^{\beta_i}-1) + (p-1)(n-1)p^{\beta_1-1} + 1 . \qquad \square$$

Liebeck gives a constructive proof. He first shows that every commutator is the product of certain "special" commutators of length at least as great. Next he exhibits a non-trivial special commutator of the required ~~maximal~~ length $c(W)$. Lastly he shows all longer special commutators are trivial. With the hypotheses of Theorem 2.22 ,

<u>2.23 COROLLARY</u> ([9])

A non-trivial special commutator of maximal length $c(W)$ is

$$[f_1, {}_{q-1}b_1, {}_{p^{\beta_2}-1}b_2, {}_{p^{\beta_3}-1}b_3, \dots , {}_{p^{\beta_m}-1}b_m]$$

where $f_1 \in A_1$, $|f_1| = p^n$ and

$$q = p^{\beta_1-1}\{ p + (p-1)(n-1)\} . \qquad \square$$

The notion of a "special" commutator generalises immediately to the permutational wreath product :

<u>2.24 DEFINITION</u>

Let A be a group and let "Λ , B" be a pair. Let $W = A\mathrm{wr}^{\Lambda}B$. Then a <u>special commutator</u> with respect to $b_1, \dots , b_m$ in B is a simple commutator whose first entry f is in A_λ for some λ in Λ , and all other entries are from $\{ b_1, \dots , b_m \}$, possibly with repetitions. $\square$

An important part of Liebeck's proof consists of looking at special commutators of the form $[f_1, {}_s b]$, where $f_1 \in A_1$ and $b \in B$, in the wreath product $A \text{ wr } B$. The results obtained generalise immediately to the permutational wreath product, and are given in this more general form for later use.

2.25 LEMMA (from Lemma 3.1 [9] , with a minor correction)

Let A be a group and let "Λ , B" be a pair. Let $W = A\mathrm{wr}^{\Lambda} B$. For λ in Λ and b in B let $f_{\lambda b^i}$ be the element in the coordinate subgroup $A_{\lambda b^i}$ corresponding to f in A . Then

$$[f_\lambda, {}_s b]^{(-1)^s} = f_\lambda f_{\lambda b}^{-\binom{s}{1}} f_{\lambda b^2}^{\binom{s}{2}} \cdots f_{\lambda b^i}^{(-1)^i\binom{s}{i}} \cdots f_{\lambda b^s}^{(-1)^s\binom{s}{s}} .$$

Note that s may exceed the order of b , and that b , or a non-trivial power of b , may act trivially on λ .

This result is easily proved by induction. □

2.26 DEFINITION

Let the conditions of Lemma 2.25 hold, and define $\mu(\lambda, s, t)$ as the coefficient of $f_{\lambda b^t}$ in $[f_\lambda, {}_s b]$ for $0 \leq t \leq s$. □

2.27 LEMMA (from 4.2 [9])

Let the conditions of Lemma 2.25 hold. Let b in B be of order p^k , and suppose there exists λ in Λ such that $\lambda b^i \neq \lambda$ for $0 < i < p^k$. Then for $0 \leq t < p^k$,

$$(-1)^{t+s}\mu(\lambda, s, t) = \begin{cases} \binom{s}{t} - \binom{s}{t+p^k} + \cdots + (-1)^j\binom{s}{t+jp^k} + \cdots & \text{for } p \text{ odd} \\ \binom{s}{t} + \binom{s}{t+2^k} + \cdots + \binom{s}{t+j2^k} + \cdots & \text{for } p=2 . \end{cases}$$

This result follows easily from Lemma 2.25 . □

The standard wreath product version of the following result was used by Shield in the proof of Theorem 2.2 : see § 5 of [17] .

<u>2.28 THEOREM</u> (from Theorem 4.3 [9])

Let A be a group and let "Λ , B" be a pair. Let $f_{\lambda'}$ be the element in $A_{\lambda'}$ corresponding to f in A for all λ' in Λ . Let b in B be of order p^k and suppose there exists λ in Λ such that $\lambda b^i \neq \lambda$ for $0 < i < p^k$. Let $q = p^{k-1}\{ p + (p-1)(n-1)\}$ for n in $\mathbb{Z}^+$. Then

i) p^n divides $\mu(\lambda,s,t)$ when $s \geq q$;

ii) p^n does not divide $\mu(\lambda,s,t)$ when $s = q-1$. □

Part ii) provides the proof that if a standard wreath product A wr B is nilpotent, then A must be of finite exponent, or we could obtain a non-trivial commutator of the form $[f_{\lambda}, {}_s b]$ for any s in $\mathbb{N}$: this can easily be seen from

<u>2.29 COROLLARY</u> (from Corollary 4.4 [9])

Let the conditions of Theorem 2.28 hold. Let f in A have order p^n , b in B have order p^k , and suppose there exists λ in Λ such that $\lambda b^i \neq \lambda$ for $1 \leq i < p^k$. Then for s in $\mathbb{Z}^+$,

i) $[f_{\lambda}, {}_s b] = 1$ for $s \geq p^{k-1}\{ p + (p-1)(n-1)\}$;

ii) $[f_{\lambda}, {}_s b]$ has order p^{ℓ} where $\ell \leq n-1$ and
$p^{k-1}\{p+(p-1)(n-\ell-1)\} \leq s < p^{k-1}\{p+(p-1)(n-\ell)\}$;

iii) $[f_{\lambda}, {}_s b]$ has order p^n where $0 < s < p^k$. □

As a corollary to Theorem 2.22 and Corollary 2.29 , we have the important case

<u>2.30 COROLLARY</u> ([9])

The nilpotency class of C_{p^n} wr C_{p^r} is $p^{r-1}\{ p + (p-1)(n-1)\}$, and a non-trivial special commutator of maximal length is given by

$$[f_1 , {}_{q-1}b]$$

where f_1 is in the first coordinate subgroup of $C_{p^n}{}^{\{C_{p^r}\}}$, $|f_1| = p^n$, $C_{p^r} = \langle b \rangle$, and $q = p^{r-1}\{ p + (p-1)(n-1)\}$. □

As observed by P.Hall on p.181 of [6] , the class of C_{p^n} wr C_{p^r} had been determined elsewhere before.

We now generalise results obtained by P.M.Neumann in [13] on the centre of a standard wreath product to the permutational wreath product. We first note that if "Λ , B" is a trivial pair , and A is a group, then $A \text{ Wr}^{\Lambda} B = \text{Cr } A^{\Lambda} \times B$ and $A \text{wr}^{\Lambda} B = \text{Dr } A^{\Lambda} \times B$ so that $Z(A \text{ Wr}^{\Lambda} B) = \text{Cr}(Z(A))^{\Lambda} \times Z(B)$ and

$$Z(A \text{ wr}^{\Lambda} B) = \text{Dr}(Z(A))^{\Lambda} \times Z(B) .$$

Thus we need consider only non-trivial pairs.

<u>2.31 LEMMA</u> (Lemma 3.3 [13])

Let G be a split extension of A by B , i.e. $A \lhd G$, B is a subgroup of G , $A \cap B = \langle 1 \rangle$ and $G = AB$. Let C be the intersection of A with the centraliser of B in G . Then the normaliser of B in G is $C \times B$, and its centraliser is $C \times Z(B)$. □

We generalise Corollary 3.4 of [13] :

2.32 THEOREM

Let A be a non-trivial group and let "Λ , B" be a non-trivial pair. Then the centre of $\underline{W} = A \ \mathrm{Wr}^{\Lambda} B$ is given by

$$Z(\underline{W}) = \{ \ \mathrm{Cr}\{Z(\mathfrak{D}_{\mathrm{Cr}} \ {}_{A}\theta) \ : \theta \text{ is an orbit of } \Lambda \ \} \ \} \times \{\mathrm{St}_B(\Lambda) \cap Z(B)\}$$

and the centre of $W = A \mathrm{wr}^{\Lambda} B$ is given by

$$Z(W) = \{ \ \mathrm{Dr}\{Z(\mathfrak{D}_{\mathrm{Dr}} \ {}_{A}\theta) \ : \theta \text{ is an orbit of } \Lambda \ \} \ \} \times \{\mathrm{St}_B(\Lambda) \cap Z(B)\}.$$

Proof

Let $fb \in Z(\underline{W})$, where $f \in \mathrm{Cr} \ A^{\Lambda}$, and $b \in B$. Then by Lemma 2.31 ,

$$fb \in C_{\underline{W}}(B) = \{C_{\underline{W}}(B) \cap \mathrm{Cr} \ A^{\Lambda}\} \times Z(B) ,$$

$$= \{ \ \mathrm{Cr}\{\mathfrak{D}_{\mathrm{Cr}} \ {}_{A}\theta \ : \theta \text{ is an orbit of } \Lambda \ \} \ \} \times Z(B)$$

by Lemma 2.19 . ...(1)

Furthermore,

$$[fb, \ g] = 1 \qquad \forall \ g \in \mathrm{Cr} \ A^{\Lambda} ,$$

$$\iff [f, \ g]^b \ [b, \ g] = 1 \qquad \forall \ g \in \mathrm{Cr} \ A^{\Lambda} , \qquad ....(2)$$

by Lemma 1.1 i) .

In particular, taking $g = g_{\lambda} \in A_{\lambda} \setminus \langle 1 \rangle$, for some λ in Λ ,

$[f_{\lambda}, \ g_{\lambda}]^b \ [b, \ g_{\lambda}] = 1$ where f_{λ} is the λth coordinate of f.

Now $[f_{\lambda}, \ g_{\lambda}] \in A_{\lambda}$, so $[f_{\lambda}, \ g_{\lambda}]^b \in A_{\lambda b}$, and

$$[b, \ g_{\lambda}] = (g_{\lambda}^{-1})^b \ g_{\lambda} = g_{\lambda b}^{-1} \ g_{\lambda} ,$$

where $g_{\lambda b} \in A_{\lambda b}$ corresponds to the same element in A as g_{λ} .

Thus $g_{\lambda} = g_{\lambda b}[f_{\lambda}, \ g_{\lambda}]^b \in A_{\lambda b}$ and so $\lambda b = \lambda$. The choice of $\lambda \in \Lambda$ was arbitrary, and so $fb \in Z(\underline{W}) \Rightarrow b \in \mathrm{St}_B(\Lambda)$.

Hence from (2) , $[f, g] = 1 \quad \forall g \in \mathrm{Cr}\, A^{\Lambda}$,

$$\Longleftrightarrow f \in Z(\mathrm{Cr}\, A^{\Lambda}) .$$

Thus from (1) , $f \in Z(\mathrm{Cr}\, A^{\Lambda}) \cap \mathrm{Cr}\{ \mathfrak{D}_{\mathrm{Cr}A\theta} : \theta \text{ is an orbit of } \Lambda \}$,

$$= \mathop{\mathrm{Cr}}_{\theta} Z(\mathfrak{D}_{\mathrm{Cr}A\theta}) ;$$

and so $Z(\underline{W}) \leq \langle \{ \mathop{\mathrm{Cr}}_{\theta} Z(\mathfrak{D}_{\mathrm{Cr}A\theta}) \} , \{ St_B(\Lambda) \cap Z(B) \} \rangle$.

For the reverse inclusion, suppose that for some $f \in \mathrm{Cr}A^{\Lambda}$ and some $b \in B$, $fb \in \langle \{ \mathop{\mathrm{Cr}}_{\theta} Z(\mathfrak{D}_{\mathrm{Cr}A\theta}) \} , \{ St_B(\Lambda) \cap Z(B) \} \rangle$.

Then for any $g \in \mathrm{Cr}A^{\Lambda}$ and any $c \in B$,

$[fb, gc] = 1$

$\Longleftrightarrow [fb, c][fb, g]^c = 1$ by Lemma 1.1 ii),

$\Longleftrightarrow [f, c]^b[b, c]([f, g]^b[b, g])^c = 1$ by Lemma 1.1 i).

Now $[f, c] = 1$ since $f \in C_{\underline{W}}(B)$, by Lemma 2.19 ,

$[b, c] = 1$ since $b \in Z(B)$,

$[f, g] = 1$ since $f \in \mathop{\mathrm{Cr}}_{\theta} Z(\mathfrak{D}_{\mathrm{Cr}A\theta}) \leq Z(\mathrm{Cr}A^{\Lambda})$,

$[b, g] = 1$ since $b \in St_B(\Lambda)$.

Hence we have the reverse inclusion, and it follows that $Z(\underline{W})$ is a direct product of $\mathop{\mathrm{Cr}}_{\theta} Z(\mathfrak{D}_{\mathrm{Cr}A\theta})$ and $St_B(\Lambda) \cap Z(B)$ since it is abelian.

The proof for W is a trivial modification of the above. $\square$

As an important corollary, which contains the result for the standard wreath product, we have

2.33 COROLLARY

Let A be a non-trivial group and let "Λ , B" be a non-trivial faithful transitive pair. Then

$$Z(A \text{ Wr}^{\Lambda} B) = Z(\mathcal{D}_{\text{Cr}_{\Lambda} A})$$

and

$$Z(A \text{ wr}^{\Lambda} B) = Z(\mathcal{D}_{\text{Dr}_{\Lambda} A}) \ . \qquad \square$$

As an example, recall from (8) of Chapter I that for $P_2 = \langle y_1, y_2 \rangle$, the diagonal is

$$\mathcal{D} = \langle y_1 \, y_1^{y_2} \, y_1^{y_2^2} \cdots y_1^{y_2^{p-1}} \rangle ,$$

which is cyclic of order p , and thus is the centre of P_2 . In general, we obtain

2.34 COROLLARY

$$Z(P_r) = \langle \prod y_1^{y_2^{k_2} y_3^{k_3} \cdots y_r^{k_r}} : \text{for } i=2,\ldots,r, \ 0 \leqslant k_i \leqslant p-1 \rangle ,$$

or if we set $s = k_r p^{r-1} + k_{r-1} p^{r-2} + \ldots + k_2 p$, with k_i as above,

$$Z(P_r) = \prod_s (s+1, s+2, \ldots , s+p) \ . \qquad \square$$

Recall that by Theorem 2.1 , the group C_p wr C_q , where p and q are distinct primes ~~not equal to 1~~ , is not nilpotent. However, it does have a centre : the diagonal group. Since C_p wr C_q is a finite group, the fact it is not nilpotent implies there exists i in $\mathbb{Z}^+$ such that

$\gamma_i(C_p \text{ wr } C_q) = \gamma_{i+j}(C_p \text{ wr } C_q)$ for all j in $\mathbb{N}$.

2.35 PROPOSITION

Let p and q be distinct primes ~~not equal to 1~~ . Then

$$\gamma_2(C_p \text{ wr } C_q) = \gamma_{2+j}(C_p \text{ wr } C_q) \quad \text{for all } j \text{ in } \mathbb{N} .$$

Proof

Since $(p, q) = 1$, there exist integers k and ℓ such that

$$kp + \ell q = 1 ,$$

for which see Lemma 1 of § 2.2 of [2] .

We define the function $\text{sgn}: \mathbb{Z} \longrightarrow \{-1, 1\}$ by

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a \in \mathbb{N} \\ -1 & \text{if } a \in \mathbb{Z}\backslash\mathbb{N}. \end{cases}$$

Let $C_p = \langle f \rangle$, and let f_λ in the λth coordinate subgroup of $C_p^{(q)}$ correspond to f in C_p for all λ in (q) . Then if $C_q = \langle b \rangle$, and $\underline{b} = b^{\text{sgn}(k)}$,

$$\begin{aligned}[f_\lambda , {}_p\underline{b}]^{(-1)^p} &= f_\lambda f_{\lambda\underline{b}}^{-\binom{p}{1}} f_{\lambda\underline{b}^2}^{\binom{p}{2}} \cdots f_{\lambda\underline{b}^{p-1}}^{(-1)^{p-1}\binom{p}{p-1}} f_{\lambda\underline{b}^p}^{(-1)^p} , \\ &= f_\lambda f_{\lambda\underline{b}^p}^{(-1)^p} , \\ &= f_\lambda f_{\nu(\lambda+\text{sgn}(k)p)}^{(-1)^p} , \text{ with } \nu \text{ as below,}\end{aligned}$$

since p divides $\binom{p}{i}$ for $1 \leqslant i \leqslant p-1$.

Hence, taking ν to be the map given in the notation section of Chapter I for the prime q , and $p \neq 2$,

$$\begin{aligned}\prod_{i=0}^{|k|-1} [f_{\nu(\text{sgn}(k)ip+1)} , {}_p b^{\text{sgn}(k)}] &= \prod_{i=0}^{|k|-1} (f_{\nu(\text{sgn}(k)ip+1)} f_{\nu(\text{sgn}(k)(i+1)p+1)}^{(-1)^p}), \\ &= f_1^{-1} f_{\nu(\text{sgn}(k)\,|k|p+1)} ,\end{aligned}$$

$$f_1^{-1} f_{\nu(\mathrm{sgn}(k)|k|p+1)} = f_1^{-1} f_{\nu(kp+1)} ,$$
$$= f_1^{-1} f_{\nu(2-\ell q)} ,$$
$$= f_1^{-1} f_2 ,$$
$$= [f_1, b] .$$

In other words, for $p \neq 2$, $[f_1, b] \in \gamma_{p+1}(C_p \text{ wr } C_q)$.

Suppose now that $p = 2$, which implies $q \geq 3$, and q is odd. Then

$$\prod_{i=0}^{\frac{q-3}{2}} [f_{1+2i}, {}_2 b] \times [f_q, {}_2 b]^{-1} = f_1 f_q [f_q, {}_2 b]^{-1}$$

since f has order 2 ,

$$= f_1 f_q f_q^{-1} f_{\nu(q+2)}^{-1} ,$$
$$= f_1 f_2^{-1} ,$$
$$= [f_1, b]^{-1} .$$

Thus for $p = 2$ we have $[f_1, b] \in \gamma_{2+1}(C_p \text{ wr } C_q)$.

The result now follows. □

2.36 COROLLARY

Let p and q be distinct primes ~~not equal to 1~~ .

i) The cpp-series for the prime p of C_p wr C_q is

$$\pi_2(C_p \text{ wr } C_q) = \pi_{2+j}(C_p \text{ wr } C_q) ,$$
$$= \langle \gamma_2(C_p \text{ wr } C_q) , C_q \rangle ,$$

for all j in $\mathbb{N}$.

ii) The cpp-series for the prime q of C_p wr C_q is

$$\pi_2(C_p \text{ wr } C_q) = \pi_{2+j}(C_p \text{ wr } C_q) = C_p^{(q)} ,$$

for all j in $\mathbb{N}$.

<u>Proof</u>

i) For any i in $\mathbb{Z}^+$, $(C_q)^{p^i} = C_q$ since $(p, q) = 1$. Hence for the prime p , $C_q \leq \pi_2(C_p \text{ wr } C_q)$. The result now follows from Proposition 2.35 .

ii) Similarly, $(C_p^{(q)})^{q^i} = C_p^{(q)}$ for all i in $\mathbb{Z}^+$, and so for the prime q ,

$$\pi_{2+j}(C_p \text{ wr } C_q) = C_p^{(q)}$$

for all j in $\mathbb{N}$, and the result now follows from Proposition 2.35 , since $\gamma_2(C_p \text{ wr } C_q) \leq C_p^{(q)}$. $\square$

CHAPTER III : Nilpotency class and cpp-class of $C_{p^n} wr^{(p^r)} P_r$, and a commutator construction.

As an immediate corollary to Shield's result, Theorem 2.2, we have

3.1 COROLLARY

Let B be a finite p-group and let A be a nilpotent p-group of class r such that for $1 \leq w \leq r$ the maximum order of a commutator of weight w in A is $p^{s(w)}$. Then

i) $c(A \text{ wr } B) = \max\{ c(C_p \text{wr } B)(w-1) + c(C_{p^{s(w)}} \text{wr } B) : 1 \leq w \leq r \}$;

ii) $c(A \text{ wr } B) = \max\{ c(C_p \text{wr } B)w + (p-1)d(B)(s(w)-1) : 1 \leq w \leq r \}$.

Proof

By Theorem 2.2, $c(C_{p^n} \text{wr } B) = a(B) + (p-1)d(B)(n-1)$, and in particular, $c(C_p \text{ wr } B) = a(B)$. The results now follow. □

Note that Corollary 3.1 i) avoids mention of the cpp-series of B.

We make the following conjecture, which generalises Corollary 3.1. We will give proofs of results towards this conjecture in Chapter VII.

3.2 CONJECTURE

Let A be a group and let "A, B" be a faithful transitive pair. If $A \text{wr}^{\Lambda} B$ is nilpotent then

i) $c(A \text{wr}^{\Lambda} B) = \max\{ c(C_p \text{wr}^{\Lambda} B)(w-1) + c(C_{p^{s(w)}} \text{wr}^{\Lambda} B) : 1 \leq w \leq r \}$;

ii) $c(A \text{wr}^{\Lambda} B) = \max\{ c(C_p \text{wr}^{\Lambda} B)w + (p-1)d(B)(s(w)-1) : 1 \leq w \leq r \}$;

where a commutator of weight w in A has order at most $p^{s(w)}$.

As stated in Remark 2.8 , Theorem 2.3 shows us that in order to find a formula for the nilpotency class of a nilpotent wreath product $A \text{ wr}^{\Lambda} B$, it is sufficient to look for a formula for the nilpotency class of $A \text{ wr}^{\Lambda} B$ where "Λ , B" is a faithful transitive pair, and by Remark 2.13 , this implies for B a finite p-group that $B \leq P_r$ and $\Lambda = (p^r)$ for some r . If Conjecture 3.2 i) is true we can restrict further to finding a formula for just $c(C_{p^n} \text{wr}^{(p^r)} B)$, and if Conjecture 3.2 ii) is true we can restrict to $c(C_p \text{wr}^{(p^r)} B)$ and $d(B)$.

We also note the following : if A is abelian of exponent p^n and "(p^r) , B" is a faithful transitive pair, then

$$c(A \text{ wr}^{(p^r)} B) = c(C_{p^n} \text{wr}^{(p^r)} B) ,$$

since if $m < n$ we can embed $C_{p^m} \text{wr}^{(p^r)} B$ in $C_{p^n} \text{wr}^{(p^r)} B$, and by Lemma 2.6 i), iii) , for i in $\mathbb{Z}^+$,

$$\gamma_i(A \text{ wr}^{(p^r)} B) = \Pi \{ [(C_{p^m})^{(p^r)}, {}_{i-1} C_{p^m} \text{wr}^{(p^r)} B] \} \gamma_i(B)$$

where the product is taken over all subgroups C_{p^m} in the decomposition of A into a direct product of finite cyclic subgroups.

Clearly we also have if "(p^r) , B" is a faithful transitive pair then B is a subgroup of P_r , and so

$$c(C_{p^n} \text{wr}^{(p^r)} B) \leq c(C_{p^n} \text{wr}^{(p^r)} P_r) .$$

We will see shortly that

3.3 THEOREM

The nilpotency class of $C_{p^n} \mathrm{wr}^{(p^r)} P_r$ is $p^{r-1}\{p+(p-1)(n-1)\}$,

which is a corollary to the more general

3.4 THEOREM

Let $r \in \mathbb{N}$, $r \geq 2$. For $i = 1, \ldots, r$, let $n_i \in \mathbb{Z}^+$. Then

$$c(C_{p^{n_1}} \mathrm{wr}\, C_{p^{n_2}} \mathrm{wr} \ldots \mathrm{wr}\, C_{p^{n_r}}) = p^{\sum_{i=2}^{r} n_i - 1}\{p+(p-1)(n_1-1)\}.$$

For the proof of these theorems we require a result about the order of commutators of a certain length, which in turn requires a result of Scott :

3.5 LEMMA (Lemma 3.6.8 [14])

Let A be a group and let "Λ, B" be a pair. Let $W = A\, \mathrm{wr}^{\Lambda} B$, and let G be a normal subgroup of A. Then $G^{\Lambda} \lhd W$ and

$$W/G^{\Lambda} \cong \{A/G\}\, \mathrm{wr}^{\Lambda} B .$$

Proof

The group G is normal in A, so G^{Λ} is normal in A^{Λ}. Furthermore, if $f \in G^{\Lambda}$, and $b \in B$, then $f^b(\lambda) \in G$ for all $\lambda \in \Lambda$ and so $f^b \in G^{\Lambda}$. Hence $G^{\Lambda} \lhd W$.

We define the map $\theta : W \longrightarrow \{A/G\}\, \mathrm{wr}^{\Lambda} B$ by $(fb)\theta = (f\theta)b$ for all b in B and all f in A^{Λ}, where $f\theta : \Lambda \longrightarrow A/G$ is the natural map $(f\theta)(\lambda) = f(\lambda)G$ for all λ in Λ. The maps $f\theta$ and θ are clearly well-defined.

We now show θ is a homomorphism. Let $f,g \in A^{\Lambda}$, and $b,c \in B$. Then,

$$(fg^{b^{-1}})\theta(\lambda) = (fg^{b^{-1}})(\lambda)G = f(\lambda)\ g(\lambda b)\ G\ ,$$
$$= (f\theta)(\lambda)\ (g\theta)(\lambda b)\ ,$$
$$= (f\theta)(g\theta)^{b^{-1}}\ (\lambda)\ ,$$

and so $(fb\ gc)\theta = (fg^{b^{-1}}bc)\theta = (fg^{b^{-1}})\theta\ bc$,

$$= (fb)\theta\ (gc)\theta\ .$$

The kernel of θ is G^{Λ} , for if $f \in A^{\Lambda}$, and $b \in B$, then

$$(fb)\theta = 1 \iff b=1 \text{ and } f\theta=1\ ,$$
$$\iff b=1 \text{ and } f(\lambda) \in G \quad \forall\ \lambda \in \Lambda\ ,$$
$$\iff fb \in G^{\Lambda}\ .$$

Lastly, θ is an epimorphism. For let $f' \in \{A/G\}^{\Lambda}$, and let $b \in B$. Let T be a transversal to G in A , and let $g \in A^{\Lambda}$ be such that

$g(\lambda) = t$ where $f'(\lambda) = tG$ for $t \in T$, and for all $\lambda \in \Lambda$.

Then g is well-defined and $g\theta = f'$, since

$$g\theta(\lambda) = g(\lambda)G = f'(\lambda) \quad \text{for all } \lambda \text{ in } \Lambda\ .$$

Hence given $f'b \in \{A/G\}\mathrm{wr}^{\Lambda}B$, there exists $gb \in W$ such that $(gb)\theta = f'b$, as required.

Hence we obtain the result. $\square$

3.6 LEMMA

Let $G_m = C_{p^m}\mathrm{wr}^{(p^r)}B$ where "(p^r) , B" is a faithful transitive pair, and let $c_m = c(G_m)$ for $m = 1, \ldots, n-1$. Let g be an element in $\gamma_{c_m+1}(G_n)$. Then g is a p^mth power, and thus of order at most p^{n-m} .

<u>Proof</u>

Let $b \in \gamma_{c(B)}(B) \setminus \gamma_{c(B)+1}(B)$. Then there exists a non-trivial commutator in $C_p wr^{(p^r)} B$ of the form $[f_1, {}_{p-1}b]$, where $1 \neq f_1 \in (C_p)_1$. Then $[f_1, {}_{p-1}b]$ is a complex commutator of length $1 + (p-1)c(B)$, and so $c_m \geq c_1 > c(B)$. Thus the element g must belong to the base group $(C_{p^n})^{(p^r)}$ of G_n . Let $\langle x \rangle = C_{p^n}$. Then $\langle x^{p^m} \rangle \lhd C_{p^n}$ and so by Lemma 3.5,

$$\begin{aligned} \{G_n / \langle x^{p^m} \rangle^{(p^r)}\} &\cong \{ \langle x \rangle / \langle x^{p^m} \rangle \} \ wr^{(p^r)} B , \\ &\cong C_{p^m} wr^{(p^r)} B , \\ &= G_m . \end{aligned}$$

Thus $g \in \langle x^{p^m} \rangle^{(p^r)}$ and so is a p^mth power. Since $(C_{p^n})^{(p^r)}$ has exponent p^n it follows that g has order at most p^{n-m} . □

In effect, Lemma 3.6 says $\gamma_{c_m+1}(C_{p^n} wr^{(p^r)} B)$ belongs to the subgroup $\{(C_{p^n})^{p^m}\} \ wr^{(p^r)} B$ of $C_{p^n} wr^{(p^r)} B$, and this subgroup is isomorphic to $C_{p^{n-m}} wr^{(p^r)} B$.

<u>Proof of Theorem 3.4</u>

We proceed by induction on r . Recall $r \geq 2$.

<u>r=2</u> : By Corollary 2.30 , $C_{p^{n_1}}$ wr $C_{p^{n_2}}$ has nilpotency class

$$p^{n_2-1}\{ p + (p-1)(n_1-1)\} \text{ , as required.}$$

Now suppose the result is true for all s such that $2 \leq s \leq r$.

Let $A_i = C_{p^{n_1}}$ wr $C_{p^{n_2}}$ wr ... wr $C_{p^{n_i}}$ for $i = 1, \ldots , r+1$.

Then by associativity of the permutational wreath product,

$$A_{r+1} = A_r \text{ wr } C_{p^{n_{r+1}}} = C_{p^{n_1}} \text{ wr}^{\Delta}\{C_{p^{n_2}} \text{ wr } C_{p^{n_3}} \text{ wr } \dots \text{ wr } C_{p^{n_{r+1}}}\}$$

where $\Delta = \{C_{p^{n_2}}\} \times \{C_{p^{n_3}}\} \times \dots \times \{C_{p^{n_{r+1}}}\}$.

By hypothesis, A_r has nilpotency class $p^{k-1}\{p+(p-1)(n_1-1)\}$,

where $k = \sum_{i=2}^{r} n_i$.

Recall from Examples 1.9 and 1.8 that $a(C_{p^{n_{r+1}}}) = p^{n_{r+1}}$ and

$d(C_{p^{n_{r+1}}}) = p^{n_{r+1}-1}$. Then by Theorem 2.2 ,

$$c(A_{r+1} = A_r \text{ wr } C_{p^{n_{r+1}}})$$

$$= \max\{\ p^{n_{r+1}}w + (p-1)p^{n_{r+1}-1}(\ s(w)-1)\ :\ 1 \leq w \leq c(A_r)\ \}\ ,$$

$$= p^{n_{r+1}-1}\max\{\ pw + (p-1)(\ s(w)-1) : 1 \leq w \leq c(A_r)\ \} \quad \dots\dots\dots(1)$$

where a commutator of weight w in A_r has order at most $p^{s(w)}$.

For $1 \leq m \leq n_1-1$ define

$$\ell_m = \max\{\ pw + (p-1)(\ s(w)-1)\ :\ p^{k-1}\{p+(p-1)(m-1)\}+1 \leq w \leq p^{k-1}\{p+(p-1)m\}\ \}\ .$$

Then by Lemma 3.6 ,

$$\ell_m \leq p.p^{k-1}\{p+(p-1)m\} + (p-1)(n_1-m-1)\ ,$$

$$= p^{k+1} + m(p-1)(p^k-1) + (p-1)(n_1-1)\ . \quad \dots\dots\dots(2)$$

Thus $\max\{\ pw + (p-1)(\ s(w)-1)\ :\ p^k+1 \leq w \leq c(A_r)\ \}$

$$= \max\{\ \ell_m\ :\ m = 1,\ \dots\ ,\ n_1-1\ \}\ ,$$

$$\leq p^k\{\ p + (p-1)(n_1-1)\ \}\ \text{ from }\ (2)\ .$$

Since $(C_{p^{n_1}})^{\Delta}$ has exponent p^{n_1} we have

$$\max\{\, pw + (p-1)(\, s(w)-1) \;:\; 1 \leqslant w \leqslant p^k \,\}$$

$$\leqslant p.p^k + (p-1)(n_1-1)\ ,$$

$$< p^k\{\, p + (p-1)(n_1-1)\ \}\ ,$$ i.e. the value of (2) for n_1-1 .

Hence from (1),

$$c(A_{r+1}) \leqslant p^{n_{r+1}-1}.p^k\{\, p + (p-1)(n_1-1)\}\ ,$$

$$= p^{n_{r+1}+k-1}\{\, p + (p-1)(n_1-1)\}\ ,$$

but this value is attained for $w = c(A_r)$ in (1) , since by Lemma 3.6 , $s(\, c\,(A_r)) = 1$. Hence we have the result. $\square$

<u>Proof of Theorem 3.3</u>

This theorem is a special case of Theorem 3.4 with $n_1 = n$ and $n_2 = n_3 = \ldots = n_{r+1} = 1$, since

$$C_{p^n} wr^{(p^r)} P_r = C_{p^n}\ wr\ \underbrace{C_p\ wr\ \ldots\ wr\ C_p}_{r\ C_p\text{'s}}\ . \qquad \square$$

<u>3.7 COROLLARY</u>

Let "(p^r) , B" be a faithful pair, and let A be an abelian group of exponent p^n . Then

$$c(A\ wr^{(p^r)}B) \leqslant p^{r-1}\{p+(p-1)(n-1)\}\ . \qquad \square$$

Note that we could have expected the nilpotency class of $C_{p^n} wr^{(p^r)} P_r$ to be at least $p^{r-1}\{p+(p-1)(n-1)\}$. For recall that by Corollary 2.30 , $p^{r-1}\{p+(p-1)(n-1)\}$ is exactly the nilpotency class of $C_{p^n}\ wr\ C_{p^r}$, and by the following result of P.Hall ,

P_r contains an element of order p^r and so we can embed C_{p^n} wr C_{p^r} in a natural way in $C_{p^n}\mathrm{wr}^{(p^r)}P_r$. For, if $\langle b \rangle$ is the top group of C_{p^n} wr C_{p^r} and y is an element of order p^r in P_r , then we can identify $(C_{p^n})^{(p^r)}$ with $(C_{p^n})^{\langle b \rangle}$ by identifying i in (p^r) with b^{i-1} in $\langle b \rangle$. Then the map θ from C_{p^n} wr C_{p^r} to $C_{p^n}\mathrm{wr}^{(p^r)}P_r$ given by $(fb^i)\theta = fy^i$, for $i \in (p^r)$ and $f \in (C_{p^n})^{(p^r)=\langle b \rangle}$, is an embedding of C_{p^n} wr C_{p^r} in $C_{p^n}\mathrm{wr}^{(p^r)}P_r$.

<u>3.8 LEMMA</u> (Lemma 6 [6])

Let A be a group containing an element of order m , and let B be a group containing an element of order n . Then A wr B contains an element of order mn . $\square$

We also find that the cpp-classes of $C_{p^n}\mathrm{wr}^{(p^r)}P_r$ and C_{p^n} wr C_{p^r} coincide, for which we require

<u>3.9 THEOREM</u> (Corollary 5.5 [17])

Let B be a finite p-group, and let A be a nilpotent p-group of class r , such that for $1 \leq w \leq r$ the maximum order of a commutator of weight w in A is $p^{s(w)}$. Then A wr B is cpp-nilpotent of cpp-class

$$d(A \text{ wr } B) = \max\{\, a(B)\, w\, p^{s(w)-1} : 1 \leq w \leq r \,\} ,$$
$$= a(B)\, d(A) \quad . \qquad \square$$

Note that there is a misprint in [17] : from Corollary 1.3 we have $d(A) = \max\{ wp^{s(w)-1} : 1 \leq w \leq r \}$,

<u>not</u> $\max\{ wp^{s(w)} : 1 \leq w \leq r \}$.

<u>3.10 THEOREM</u>

Let $A_r = C_{p^{n_1}} \text{ wr } C_{p^{n_2}} \text{ wr } \cdots \text{ wr } C_{p^{n_r}}$, where $n_i \in \mathbb{Z}^+$ for $i = 1, \ldots, r$. Then

$$d(A_r) = p^{\sum_{i=1}^{r} n_i - 1} .$$

<u>Proof</u>

From Examples 1.9 and 1.8 , $a(C_{p^n}) = p^n$ and $d(C_{p^n}) = p^{n-1}$.

Hence from Theorem 3.9 ,

$$d(C_{p^{n_1}} \text{ wr } C_{p^{n_2}}) = p^{n_1+n_2-1} ,$$

i.e. the base case of an induction argument on r .

(Note we could expect $d(C_{p^{n_1}} \text{ wr } C_{p^{n_2}}) \geq p^{n_1+n_2-1}$, since by Lemma 3.8, $C_{p^{n_1}} \text{ wr } C_{p^{n_2}}$ contains an element of order $p^{n_1+n_2}$, and $d(C_{p^{n_1+n_2}}) = p^{n_1+n_2-1}$.)

Now suppose the result is true for $r-1$, so $d(A_{r-1}) = p^{\sum_{i=1}^{r-1} n_i - 1}$.

Then by associativity of the permutational wreath product,

$$\begin{aligned} d(A_r) &= d(A_{r-1} \text{ wr } C_{p^{n_r}}) , \\ &= p^{\sum_{i=1}^{r-1} n_i - 1} . p^{n_r} \quad \text{by Theorem 3.9 ,} \\ &= p^{\sum_{i=1}^{r} n_i - 1} . \end{aligned}$$

The result now follows by induction. ▯

<u>3.11 COROLLARY</u>

$$d(C_{p^n} \text{wr}^{(p^r)} P_r) = p^{n+r-1} .$$ ▯

As a corollary to Theorem 3.3 and Theorem 2.2 we have bounds for Shield's constants a and d :

3.12 COROLLARY

Let B be a group of order p^r . Then

i) $1 + (p-1)r \leqslant a(B) \leqslant p^r$;

and ii) $1 \leqslant d(B) \leqslant p^{r-1}$;

furthermore, these bounds are best possible given only the order of B .

Proof

i) By definition,

$$a(B) = 1 + (p-1)\Sigma\{ n_i : 1 \leqslant i \leqslant d(B) \} ,$$

where $|\pi_i(B)| = p^{n_i}$. Hence the minimum possible value is attained when B is an elementary abelian group, i.e.

$$a(B) \geqslant 1 + (p-1)r .$$

By Theorem 2.2 ,

$$c(C_p \text{wr } B) = a(B) .$$

By Lemma 2.6 i) ,

$$\gamma_n(C_p \text{wr } B) = [(C_p)^B, {}_{n-1}(C_p \text{wr } B)]\, \gamma_n(B) \quad \text{for } n \in \mathbb{Z}^+.$$

Since $C_p \text{wr } B$ is nilpotent, it follows that for $n \in \mathbb{Z}^+$,and $n < c(C_p \text{wr } B) = a(B)$,

$$[(C_p)^B, {}_{n+1}(C_p \text{wr } B)] < [(C_p)^B, {}_n(C_p \text{wr } B)] ,$$

and so since $|(C_p)^B| = p^{p^r}$, $[(C_p)^B, {}_{a(B)}(C_p \text{wr } B)] = \langle 1 \rangle$,

$$[(C_p)^B, {}_n(C_p \text{wr } B)] = \langle 1 \rangle \quad \text{for } n \geqslant p^r .$$

Thus $a(B) = c(C_p \text{wr } B) \leqslant \max\{ p^r , c(B) \} = p^r$.

As we saw in Example 1.9, $a(C_{p^r}) = p^r$, and so this result is best possible for the order of B.

ii) The minimum possible length of the cpp-series of a group is 1, attained by any elementary abelian group. Hence $d(B) \geq 1$ is the best lower bound given only the order of B.

Now since C_{p^n} wr $B = C_{p^n}$ wr$^B B$, where "B, B" is the right regular representation of B, we may embed C_{p^n} wr B in $C_{p^n}\text{wr}^{(p^r)}P_r$. Thus by Theorem 3.3,

$$c(C_{p^n} \text{ wr } B) \leq p^{r-1}\{ p + (p-1)(n-1) \} .$$

But by Theorem 2.2,

$$\begin{aligned} c(C_{p^n} \text{ wr } B) &= \max\{ a(B)w + (p-1)d(B)(s(w)-1) : 1 \leq w \leq c(C_{p^n})\} \\ &= a(B) + (p-1)d(B)(n-1) . \end{aligned}$$

Hence $(p-1)(n-1)d(B) \leq p^{r-1}\{p+(p-1)(n-1)\} - a(B)$,

$$< p^{r-1}\{p+(p-1)(n-1)\}.$$

Thus $d(B) < \dfrac{p^r}{(p-1)(n-1)} + p^{r-1}$.

Let n tend to infinity. Then $d(B) \leq p^{r-1}$. We saw in Example 1.8 that $d(C_{p^r}) = p^{r-1}$, and so this bound is best possible given only the order of B. $\square$

We now mention other bounds on the nilpotency class of C_{p^n} wr$^{(p^r)}B$ where "(p^r), B" is a faithful transitive pair. By Corollary 2.15, we can embed C_{p^n} wr$^{(p^r)}B$ in C_{p^n} wr B, and thus the nilpotency class of the former is bounded by that of the latter, i.e. by the class of a standard wreath product. This same result is obtained by Shield in the more general Theorem 4.6 [17]. Note that for the faithful transitive pair "(p^r) , C_{p^r}" , which is equal to "C_{p^r} , C_{p^r}" , this upper bound coincides with that of Corollary 3.7 , and is attained , since by Corollary 2.30 and Theorem 3.3 ,

$$c(C_{p^n} \text{ wr } C_{p^r}) = p^{r-1}\{p+(p-1)(n-1)\} = c(C_{p^n}\text{wr}^{(p^r)}P_r) .$$

For a lower bound, note that if B is of exponent p^k then $C_{p^n}\text{wr}^{(p^r)}B$ contains a subgroup isomorphic to C_{p^n} wr C_{p^k} , by the same argument as on p.67 before Lemma 3.8 , and so by Corollary 2.30 ,

<u>3.13 PROPOSITION</u>

Let "(p^r) , B" be a faithful transitive pair such that B has exponent p^k . Then

$$c(C_{p^n}\text{wr}^{(p^r)}B) \geqslant p^{k-1}\{ p + (p-1)(n-1) \} . \qquad \square$$

<u>3.14 COROLLARY</u>

Let "(p^r) , B" be a faithful transitive pair, and let B have exponent p^r . Then

$$c(C_{p^n}\text{wr}^{(p^r)}B) = p^{r-1}\{ p + (p-1)(n-1) \} . \qquad \square$$

It comes as a surprise that the nilpotency class - and the cpp-class - of $C_{p^n}\text{wr}^{(p^r)}P_r$ is only that of its comparatively small subgroup C_{p^n} wr C_{p^r} : the order of $C_{p^n}\text{wr}^{(p^r)}P_r$ is

$$(p^n)^{p^r}.p^{p^{r-1}+p^{r-2}+\,\dots\,+1} ,$$

whereas the order of C_{p^n} wr C_{p^r} is $(p^n)^{p^r}.p^r$.

For this reason, the non-trivial commutators of maximal length $c(C_{p^n} wr^{(p^r)} P_r)$ of the form $[f_\lambda, {}_{q-1}b]$, where

$$q = c(C_{p^n} wr^{(p^r)} P_r) = c(C_{p^n} \text{ wr } C_{p^r}) = p^{r-1}\{p+(p-1)(n-1)\}$$

and b is of order p^r , are not very illuminating. We will construct a different non-trivial simple commutator of maximal length which perhaps reflects the structure of

$$C_{p^n} wr^{(p^r)} P_r \cong (C_{p^n} wr^{(p^{r-1})} P_{r-1}) \text{ wr } C_p$$

more fully, since it is built up by recurrence on r . This will lead to the construction of the lower central and cpp-series of $C_{p^n} wr^{(p^r)} P_r$ in Chapter IV . The ideas on which the construction is based will also come in use later in finding the class of $C_{p^n} wr^{(p^2)} B$ and in finding an alternative proof of Theorem 3.3 which does not use Shield's result, Theorem 2.2 , at all .

We first establish some notation. Recall that for $1 \leqslant s < r$ we identify P_s with the subgroup $\langle y_1, \dots , y_s \rangle$ of P_r . We can extend this embedding in a natural way to an embedding of $C_{p^n} wr^{(p^s)} P_s$ in $C_{p^n} wr^{(p^r)} P_r$, under which the base group $(C_{p^n})^{(p^s)}$ of $C_{p^n} wr^{(p^s)} P_s$ is identified with the subgroup $(C_{p^n})^{(p^s)}$ of $C_{p^n} wr^{(p^r)} P_r$ . ...(3)

From now on, for the chapter, let $\langle f_1 \rangle$ be the first coordinate subgroup of $(C_{p^n})^{(p^r)}$, and let $f_\lambda \in (C_{p^n})_\lambda$ be the conjugate of f_1 in $(C_{p^n})_\lambda$, so $\langle f_\lambda \rangle = (C_{p^n})_\lambda$ and

$$f_{\ell_1+\ell_2 p+\dots+\ell_r p^{r-1}}$$

$$= f_{(1)(y_1^{\ell_1-1}\, y_2^{\ell_2} \dots y_r^{\ell_r})} ,$$

$$= f_1^{\,y_1^{\ell_1-1}\, y_2^{\ell_2} \dots y_r^{\ell_r}} ,$$

where for $j = 2, \dots, r$, $0 \leq \ell_j \leq p-1$,

and for $j = 1$, $1 \leq \ell_1 \leq p$ . ...(4)

3.15 DEFINITION

We now define our commutators $\{ g_i \}$, where g_i is a non-trivial simple commutator of maximal length in $C_{p^n} wr^{(p^i)} P_i$, and is identified with its image under the embedding of $C_{p^n} wr^{(p^i)} P_i$ in $C_{p^n} wr^{(p^r)} P_r$ where $i < r$.

For $i = 1$ we define

$$g_1 = [\, f_1 \,,\, {}_{n(p-1)}y_1 \,] .$$

Note that g_1 is precisely the commutator given by Corollary 2.30 for $r = 1$.

Now suppose we have defined $g_1, \dots, g_{r-1}$. Then g_r is the simple commutator obtained by inserting $(p-1)$ y_r's between each pair of terms in the simple commutator g_{r-1} , with a further $(p-1)$ y_r's added at the end. For example,

$$g_2 = [\, f_1 \,,\, {}_{n(p-1)}(\,{}_{p-1}y_2 \,,\, y_1) \,,\, {}_{p-1}y_2 \,] . \qquad \square$$

Note that g_r is a commutator p times as long as g_{r-1} . Hence, since the length of g_1 is $1+n(p-1) = p+(p-1)(n-1)$, the length of $\underline{g}_r$ is $p^{r-1}\{ p + (p-1)(n-1) \} = c(C_{p^n} wr^{(p^r)} P_r)$, as required.

We will show that

i) there exist non-trivial commutators of maximal length $c(C_{p^n} wr^{(p^r)} P_r)$ in $C_{p^n} wr^{(p^r)} P_r$ of the form $[f_1, b_1, \ldots, b_s]$, where $b_1, \ldots, b_s$ are of nilpotency weight 1 and $\langle f_1 \rangle$ is the first coordinate subgroup of $(C_{p^n})^{(p^r)}$;

ii) the $\underline{g}_i$'s , which are of the form given in i) , are non-trivial and in the centre of their respective groups $C_{p^n} wr^{(p^i)} P_i$;

iii) that from the construction of the $\underline{g}_i$'s it will then seem plausible that $\underline{g}_i$ is a commutator of maximal length in $C_{p^n} wr^{(p^i)} P_i$.

We generalise Liebeck's notion of a special commutator - see Definition 2.24 :

<u>3.16 DEFINITION</u>

Let A be a group and let "(p^r) , B" be a faithful transitive pair. Then a simple commutator in $A\ wr^{(p^r)} B$ is an <u>extra-special commutator</u> (<u>e.s.c.</u>) if it is of the form

$$[h_1, b_1, \ldots, b_s]$$

where h_1 is in the first coordinate subgroup of $A^{(p^r)}$ and the nilpotency weight of each b_i is 1 , where $b_i \in B$ for $i = 1, \ldots, s$. ▯

The following is a generalisation of Lemma 5.4 of [9] :

3.17 LEMMA

Let $W = A \operatorname{wr}^{\Lambda} B$, where A is an abelian group and "Λ, B" is a pair. Then

i) if $g_i \in A_i$ and $b \in B$ then $[b, g_i] = [g_i^{-1}, b]$;

ii) if $f, g \in A^{\Lambda}$ and $w \in W$ then

$$[f, wg] = [f, w]$$

and

$$[fg, w] = [f, w][g, w] ;$$

iii) if $w_i = h_i b_i$ where $h_i \in A^{\Lambda}$ and $b_i \in B$, and if $f, g \in A^{\Lambda}$, then

$$[f, w_1, \ldots, w_s] = [f, b_1, \ldots, b_s] ,$$

and

$$[fg, w_1, \ldots, w_s] = [f, w_1, \ldots, w_s][g, w_1, \ldots, w_s] .$$

Proof

The proofs are essentially those of Liebeck .

i) $[b, g_i] = b^{-1}g_i^{-1}bg_i = (g_i^{-1})^b g_i$,

$= g_i(g_i^{-1})^b$ since A^{Λ} is abelian,

$= [g_i^{-1}, b]$.

ii) The group A^{Λ} is abelian and is a normal subgroup of W. Now apply Lemma 1.1 ii) , i) .

iii) First note that since $A^{\Lambda} \lhd W$, we have

$$[A^{\Lambda}, W] \leqslant A^{\Lambda} ,$$

and so by induction, $[A^{\Lambda}, {}_{n}W] \leq A^{\Lambda}$.

We now proceed by induction on s to prove both parts. The case $s = 1$ is dealt with by ii).

Suppose we have shown $[f, w_1, \dots, w_{s-1}] = [f, b_1, \dots, b_{s-1}]$.
Then since $[f, w_1, \dots, w_{s-1}] \in A^{\Lambda}$,

$$\begin{aligned}[f, w_1, \dots, w_s] &= [[f, w_1, \dots, w_{s-1}], w_s], \\ &= [[f, b_1, \dots, b_{s-1}], w_s], \\ &= [[f, b_1, \dots, b_{s-1}], b_s] \quad \text{by ii)},\end{aligned}$$

and we have the first part by induction.

Now suppose we have shown

$$[fg, w_1, \dots, w_{s-1}] = [f, w_1, \dots, w_{s-1}][g, w_1, \dots, w_{s-1}].$$

Then since $[f, w_1, \dots, w_{s-1}], [g, w_1, \dots, w_{s-1}] \in A^{\Lambda}$,

$$\begin{aligned}[fg, w_1, \dots, w_s] &= [[f, w_1, \dots, w_{s-1}][g, w_1, \dots, w_{s-1}], w_s], \\ &= [[f, w_1, \dots, w_{s-1}], w_s][[g, w_1, \dots, w_{s-1}], w_s]\end{aligned}$$

by ii), as required for the induction. □

3.18 COROLLARY

Let $W = A \operatorname{wr}^{(p^r)} B$, where A is an abelian group and "(p^r), B" is a faithful transitive pair. Let $[w_1, \dots, w_s]$ be a commutator in W. Then $[w_1, \dots, w_s]$ can be expressed in the form $\underline{f}\,\underline{b}$ where $\underline{f}$ is a product of extra-special commutators each of length at least s, and $\underline{b}$ is a product of simple commutators each of length at least s in B.

Proof

The commutator $[w_1, \dots, w_s]$ is an expression for an element in $\gamma_s(W)$. By Lemma 2.6 i),

$$\gamma_s(W) = [\, A^{(p^r)}, {}_{s-1}W \,]\, \gamma_s(B) ,$$

so $[w_1, \dots, w_s] = \underline{g}\,\underline{b}$ where $\underline{g}$ is a product of simple commutators of the form $[g, v_1, \dots, v_{s-1}]$ with g in $A^{(p^r)}$ and $v_1, \dots, v_{s-1}$ in W, and by Lemma 1.5, $\underline{b}$ is a product of simple commutators of length at least s, since B is nilpotent.

Now $g = \prod_{i=1}^{p^r} g_i^{t_i}$ where $g_i \in A_i$ and $t_i \in \mathbb{Z}$. Note that since the g_i's belong to distinct coordinate subgroups, they commute, i.e. $\langle g_i : i = 1, \dots, p^r \rangle$ is abelian. Let $v_j = h_j b_j$ where $h_j \in A^{(p^r)}$ and $b_j \in B$ for $j = 1, \dots s-1$. Then by Lemma 3.17 iii),

$$[g, v_1, \dots, v_{s-1}] = [\, \prod_{i=1}^{p^r} g_i^{t_i}, b_1, \dots, b_{s-1}] ,$$

$$= \prod_{i=1}^{p^r} \{ \, [g_i, b_1, \dots, b_{s-1}]^{t_i} \} .$$

Since "(p^r), B" is transitive, there exists $b(i)$ in B such that $1.b(i) = i$, for $i = 1, \dots, p^r$. Hence if $g(i)$ in A_1 is such that $g_i = g(i)^{b(i)} = g(i)[g(i), b(i)]$, by Lemma 3.17 iii)

$$[g_i, b_1, \dots, b_{s-1}] = [\, g(i)[g(i), b(i)], b_1, \dots, b_{s-1}] ,$$

$$= [\, g(i), b_1, \dots, b_{s-1}] \times$$

$$[\, g(i), b(i), b_1, \dots, b_{s-1}] .$$

Now write each element of B in these commutators as a product of elements of nilpotency weight 1 and expand using Lemma 1.1 ii).

3.19 LEMMA

Let $W = A \text{ wr}^{(p^r)} B$, where A is an abelian group of exponent p^n and "(p^r), B" is a faithful transitive pair. Let b in $\gamma_i(B) \setminus \gamma_{i+1}(B)$ be of order p^k. Then

$$c(A \text{ wr}^{(p^r)} B) \geq 1 + i\{ p^{k-1}\{p+(p-1)(n-1)\} - 1 \} .$$

Proof

Since B is a transitive subgroup of $\mathbf{P}_r$, using the same arguments as in the proof of Lemma 2.18, it follows that $A \text{ wr}^{(p^r)} B$ contains a subgroup isomorphic to $A \text{ wr} \langle b \rangle$. Let h_1 in A_1 be of order p^n, and let $q = p^{k-1}\{ p + (p-1)(n-1) \}$. Then by Corollary 2.30, the commutator $[h_1, {}_{q-1}b]$ is a non-trivial commutator in $A \text{ wr} \langle b \rangle$. As a commutator in $A \text{ wr}^{(p^r)} B$,

$$1 \neq [h_1, {}_{q-1}b] \in [\gamma_1(W), {}_{q-1}\gamma_i(B)] \leq \gamma_{1+i(q-1)}(W) ,$$

since $\gamma_i(B) \leq \gamma_i(W)$ and in general $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.

Hence we have the result. $\square$

3.20 COROLLARY

Let $W = A \text{ wr}^{(p^r)} B$ as in Lemma 3.19, and let $\underline{w}$ be a non-trivial commutator of maximal length $c(A \text{ wr}^{(p^r)} B)$ in $A \text{ wr}^{(p^r)} B$. Then $\underline{w}$ is the product of extra-special commutators of length $c(A \text{ wr}^{(p^r)} B)$.

Proof

By Lemma 3.19, $c(W) > c(B)$ since we may take b in $\gamma_{c(B)}(B)$ of order p. The result now follows from Corollary 3.18. $\square$

3.21 PROPOSITION

Let $W = A\ wr^{(p^r)} B$ where A is an abelian group of exponent p^n and "(p^r) , B" is a faithful transitive pair. Then there exists a non-trivial extra-special commutator of length $c(W)$ in W whose first entry $h_1 \in A_1$ is of order p^n .

Proof

By Corollary 3.20 , there exists an e.s.c. $[h_1, b_1, \ldots, b_{c(W)-1}] \neq 1$ where $h_1 \in A_1$. We want to show we may choose h_1 to have order p^n . Let h_i be the conjugate of h_1 in A_i for $i = 2, \ldots, p^r$. Then

$$[h_1, b_1, \ldots, b_{c(W)-1}] = \prod_{i=1}^{p^r} h_i^{t_i} ,$$

where the t_i's are independent of the order of h_1 . Suppose h_1 has order p^m where $m \leq n$. Then since $[h_1, b_1, \ldots, b_{c(W)-1}] \neq 1$, there exists j such that p^m does not divide t_j , and so p^n does not divide t_j . Hence if we replace h_1 in the commutator by an element of A_1 of order p^n we still obtain a non-trivial commutator, and we have the result. $\square$

We now turn to proving that the commutators $\underline{g}_i$ of Definition 3.15 are non-trivial and in the centre of their respective groups $C_{p^n} wr^{(p^i)} P_i$. Note we do not use Theorem 3.3 , as we are giving a plausibility argument for Theorem 3.3 .

3.22 PROPOSITION

Let f_i be defined as on p.72 for $C_{p^n} wr^{(p^r)} P_r$, where $i \in (p^r)$.

Then for $1 \leq m \leq n$, $j \in \{1, \ldots, r\}$, $i \in (p^j)$,

$$[f_i, {}_{m(p-1)}y_j] = \left\{\prod_{k=0}^{p-1} (f_{s_k}{}^{t_k})\right\}^{p^m} \left\{\prod_{k=0}^{p-1} f_{s_k}\right\}^{tp^{m-1}}$$

where $0 \leq t_k < p^{n-m}$, $0 < t < p$, and $s_k \equiv i + kp^{j-1} \mod p^j$.

If $i \in (p^r) \setminus (p^j)$ then

$$[f_i, {}_{m(p-1)}y_j] = 1 .$$

<u>Proof</u>

If $i \in (p^r) \setminus (p^j)$ then $[f_i, y_j] = 1$, since

$$y_j = \prod_{\ell=1}^{p^{j-1}} (\ \ell,\ p^{j-1}+\ell,\ 2p^{j-1}+\ell\ , \ldots ,\ p^j - p^{j-1}+\ell\ )$$

implies $\sigma(y_j) = (p^j)$.

Now suppose $i \in (p^j)$, so i belongs to an orbit of "$(p^r), \langle y_j \rangle$" of size p . Then $\langle f_i, y_j \rangle \cong C_{p^n}$ wr C_p . Let $G = \langle f_i, y_j \rangle$.

Then by Corollary 2.29 ,

$$[f_i, {}_{m(p-1)}y_j]$$

has order p^{n-m+1} , and so

$$[f_i, {}_{m(p-1)}y_j] = \left\{\prod_{k=0}^{p-1} f_{s_k}{}^{t_k}\right\}^{p^m} \left\{\prod_{k=0}^{p-1} f_{s_k}{}^{u_k}\right\}^{p^{m-1}}$$

where $t_k \geq 0$, $0 \leq u_k < p$ and $s_k \equiv i + kp^{j-1} \mod p^j$, and

there exists k' such that $u_{k'} > 0$ . ...(5)

Let $\theta : C_{p^n} \mathrm{wr}^{(p^r)} P_r \longrightarrow (C_{p^n})^{p^{n-m}} \mathrm{wr}^{(p^r)} P_r$ be the natural

homomorphism given by

$$(fb)\theta = f^{p^{n-m}} b \text{ where } f \in (C_{p^n})^{(p^r)} \text{ and } b \in B .$$

Under this homomorphism, $G \longrightarrow \langle f_i^{p^{n-m}}, y_j \rangle \cong C_{p^m} \text{ wr } C_p$.

In particular,

$$[f_i, {}_{m(p-1)}y_j]\theta = [f_i^{p^{n-m}}, {}_{m(p-1)}y_j] ,$$

$$\in Z(\langle f_i^{p^{n-m}}, y_j \rangle) \text{ by Corollary 2.30 ,}$$

$$= \left\langle \prod_{k=0}^{p-1} \left(f_i^{p^{n-m}}\right)^{y_j^k} \right\rangle \text{ by Corollary 2.33 ,}$$

$$= \left\langle \prod_{k=0}^{p-1} f_{s_k}^{p^{n-m}} \right\rangle .$$

Thus $\left(\left\{ \prod_{k=0}^{p-1} f_{s_k}^{t_k} \right\}^{p^m} \left\{ \prod_{k=0}^{p-1} f_{s_k}^{u_k} \right\}^{p^{m-1}} \right) \theta = \left\{ \prod_{k=0}^{p-1} f_{s_k}^{u_k} \right\}^{p^{n-1}}$ implies

that, by (5) , $u_0 = u_1 = \ldots = u_{p-1} \neq 0$, i.e.

$$\left\{ \prod_{k=0}^{p-1} f_{s_k}^{u_k} \right\}^{p^{m-1}} = \left\{ \prod_{k=0}^{p-1} f_{s_k} \right\}^{tp^{m-1}} \quad \text{where } 0 < t < p \text{ , as}$$

required, and we have the result. □

3.23 LEMMA

Let G be a group, and let x,y be elements of G such that $\langle x \rangle^{\langle y \rangle}$ is abelian, i.e. for i in $\mathbb{Z}$, the conjugates of x^i with respect to $\langle y \rangle$ commute. Then for s in $\mathbb{Z}^+$,

$$[x, {}_s y] = \left\{ x\, (x^y)^{-\binom{s}{1}} \ldots (x^{y^j})^{(-1)^j\binom{s}{j}} \ldots (x^{y^s})^{(-1)^s\binom{s}{s}} \right\}^{(-1)^s} .$$

Proof

The result follows easily by induction, as in Lemma 2.25 , or can be deduced from Lemma 2.25 using Lemma 3.17 iii) . □

3.24 PROPOSITION

Let f be an element in the base group of $C_{p^n} wr^{(p^r)} P_r$, where $P_r = \langle y_1, \dots, y_r \rangle$ as before. Then for $1 \leq \{i, j\} < k \leq r$,

$$[f, y_i, y_j] = [f, y_i, {}_{p-1}y_k, y_j]^{(-1)^{p-1}} .$$

Proof

Let $f_1, \dots, f_{p^r}$ be defined as in (4) . Then since $\sigma(y_i) = (p^i)$, $[f, y_i]$ is a product of powers of $f_1, \dots, f_{p^i}$. Let $\underline{h} = [f, y_i]$.

Then by Lemma 3.23 ,

$$[f, y_i, {}_{p-1}y_k] = [\underline{h}, {}_{p-1}y_k] ,$$

$$= \left\{ \underline{h} (\underline{h}^{y_k})^{-\binom{p-1}{1}} (\underline{h}^{y_k^2})^{\binom{p-1}{2}} \dots (\underline{h}^{y_k^{p-1}})^{(-1)^{p-1}\binom{p-1}{p-1}} \right\}^{(-1)^{p-1}} .$$

Now $\underline{h}^{y_k^\ell}$ is a product of powers of $f_1^{y_k^\ell}, f_2^{y_k^\ell}, \dots, f_{p^i}^{y_k^\ell}$, i.e. of $f_{1+\ell p^{k-1}}, f_{2+\ell p^{k-1}}, \dots, f_{p^i+\ell p^{k-1}}$. But $\sigma(y_j) = (p^j)$, where $j < k$ by hypothesis, so y_j and $\underline{h}^{y_k^\ell}$ commute for $0 < \ell < p$. Thus

$$[f, y_i, {}_{p-1}y_k, y_j]^{(-1)^{p-1}} = [\underline{h}, y_j] = [f, y_i, y_j] . \qquad \square$$

Clearly if $\lambda \in (p^k)$ we also have, by the same proof,

3.25 COROLLARY

$$[f_\lambda, {}_{p-1}y_k, y_j] = [f_\lambda, y_j]^{(-1)^{p-1}} , \text{ where } 1 \leq j < k \leq r ,$$

as above. $\square$

3.26 THEOREM

Let $\{g_i\}$ be defined as in Definition 3.15 . Then $g_i \neq 1$, and g_i is in the centre of $C_{p^n} wr^{(p^i)} P_i$ where $i \in \mathbb{Z}^+$.

Note Recall we are presenting a plausibility argument for the class of $C_{p^n} wr^{(p^r)} P_r$. Ofcourse, it follows automatically from Theorem 3.3 that g_i is central in $C_{p^n} wr^{(p^i)} P_i$.

Proof

Use Proposition 3.22 , Lemma 3.23 , Proposition 3.24 , and Corollary 3.25 to show $g_i \neq 1$.

Now by these same results, the element given by g_i can also be expressed as

$$[f_1, {}_{n(p-1)}y_1, {}_{p-1}y_2, {}_{p-1}y_3, \ldots, {}_{p-1}y_i]^{(-1)^t}, \qquad \ldots\ldots.(6)$$

for some $t \in \{0, 1\}$.

We proceed by induction on i to show that

$$[f_1, {}_{n(p-1)}y_1, {}_{p-1}y_2, \ldots, {}_{p-1}y_i] = (f_1 f_2 \ldots f_{p^i})^{up^{n-1}},$$

for some u such that $0 < u < p$. This implies that g_i is central in $C_{p^n} wr^{(p^i)} P_i$, since by Corollary 2.33 ,

$$Z(C_{p^n} wr^{(p^i)} P_i) = \left\langle \prod_{j=1}^{p^i} f_j \right\rangle .$$

$i=1$: By Proposition 3.22 ,

$$[f_1, {}_{n(p-1)}y_1] = (f_1 \ldots f_p)^{up^{n-1}}, \quad \text{for some } u \text{ such that } 0 < u < p .$$

Now suppose the result is true for $i-1$. Then

$$[f_1, {}_{n(p-1)}y_1, {}_{p-1}y_2, \ldots, {}_{p-1}y_i]$$

$$= [(f_1 \ldots f_{p^{i-1}})^{up^{n-1}}, {}_{p-1}y_i] \quad \text{by hypothesis,}$$

$$= \left(\prod_{k=1}^{p^{i-1}} [f_k, {}_{p-1}y_i]\right)^{up^{n-1}} \quad \text{by Lemma 3.17 iii) ,}$$

$$= \left(\prod_{k=1}^{p^{i-1}} \left(f_k\, f_{k+p^{i-1}}^{-\binom{p-1}{1}}\, f_{k+2p^{i-1}}^{\binom{p-1}{2}} \ldots f_{k+p^i-p^{i-1}}^{(-1)^{p-1}}\right)^{(-1)^{p-1}}\right)^{up^{n-1}}$$

by Lemma 2.25 ,

$$= \left(\prod_{k=1}^{p^{i-1}} (f_k\, f_{k+p^{i-1}} \ldots f_{k+p^i-p^{i-1}})^{(-1)^{p-1}}\right)^{up^{n-1}}, \quad \text{since}$$

$\binom{p-1}{i} \equiv (-1)^i \mod p$ and $|f_k| = p^n$,

$$= (f_1 \ldots f_{p^i})^{(-1)^{p-1} up^{n-1}} ,$$

$$= (f_1 \ldots f_{p^i})^{u' p^{n-1}} \quad \text{where } 0 < u' < p \text{ as required.}$$

Hence, by induction,

$$[f_1, {}_{n(p-1)}y_1, {}_{p-1}y_2, \ldots, {}_{p-1}y_i] = (f_1 \ldots f_{p^i})^{up^{n-1}}, \text{ where}$$

$0 < u < p$, and so g_i is central in $C_{p^n} \mathrm{wr}^{(p^i)} P_i$, as required.

$\square$

Note that since the order of each y_i is p , so $f_1^{y_i^p} = f_1$,

$$[f_1, {}_p y_i] = \left(f_1\, f_{p^{i-1}+1}^{-\binom{p}{1}}\, f_{2p^{i-1}+1}^{\binom{p}{2}} \ldots f_1^{(-1)^p}\right)^{(-1)^p}$$

by Lemma 2.25 . This is clearly a p-th power if p is odd, since p divides $\binom{p}{j}$ for $1 \leq j \leq p-1$, and the exponent of f_1 is $1 + (-1)^p = 0$. If $p = 2$ we have the exponent of f_1 is $1 + (-1)^2 = 2$ and $\binom{2}{1}$ is divisible by 2 , so $[f_1, {}_2y_i]$ is a

2nd power. In the construction of the g_i's from g_1, no more than $(p-1)$ of any y_j, $j=1, \ldots, i$, are consecutive, so that each g_i is the longest possible non-trivial simple commutator in $f_1, y_1, \ldots, y_i$, containing exactly $n(p-1)$ y_1's. For the particular case $i=1$, the commutator g_1 is known from Liebeck's result Corollary 2.30 to be of maximal length.

When $n=1$, the g_i's are non-trivial commutators of maximal length in the Sylow-p-subgroups of the symmetric groups, P_i, where "f_1" $= y_1$ and "y_j" $= y_{j+1}$ in the usual description of P_i as $\langle y_1, \ldots, y_i \rangle$. In fact, Proposition 3.24 gives us a way of constructing a non-trivial commutator in P_{r+1} which is p times as long as a given non-trivial simple commutator in P_r.

The g_i's have a further use: in describing the lower central and cpp-series of $C_{p^n} \mathrm{wr}^{(p^r)} P_r$, which is the subject of the next chapter.

CHAPTER IV : The lower central and cpp- structure of $C_{p^n} \operatorname{wr}^{(p^r)} P_r$, and the lower central structure of $C_{p^n} \operatorname{wr} C_{p^r}$.

Note we will sometimes use $Q_{n,r}$ to denote $C_{p^n} \operatorname{wr}^{(p^r)} P_r$, and D to denote the base group $(C_{p^n})^{(p^r)}$ of $Q_{n,r}$. For $i = 1, \dots, r$, D_i will denote the base group of $\langle y_i, \dots, y_r \rangle \cong C_p \operatorname{wr}^{(p^{r-i})} P_{r-i}$.
Note that $Q_{n,r} = DD_1 \dots D_r$, and $P_r = D_1 \dots D_r$ . ...(1)

We will generalise the diagram of P_r given by Weir in [18] to a diagram of $Q_{n,r}$. We begin by describing the diagram for P_r , using the notation established in Chapter I .

$\underline{P_r}$:

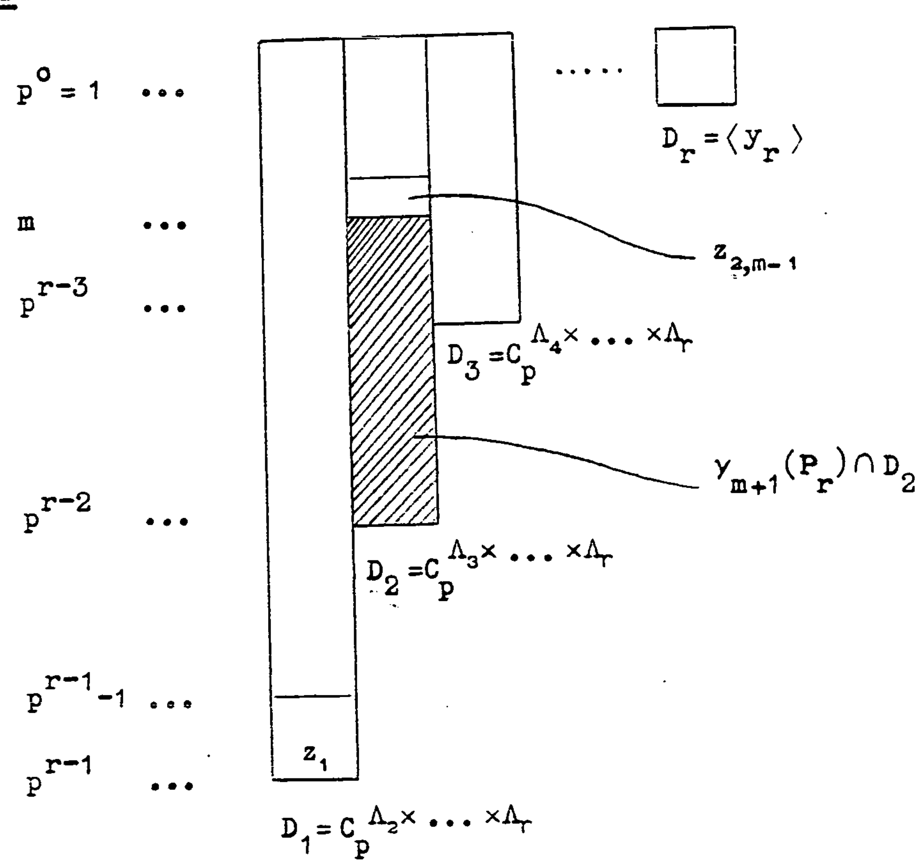


Fig. 3

Notice that the base group D_1 of $P_r \cong C_p \operatorname{wr}^{(p^{r-1})} P_{r-1}$ has order $p^{p^{r-1}}$, while the nilpotency class of P_r is p^{r-1}, by Theorem 3.3, since $C_p \operatorname{wr}^{(p^{r-1})} P_{r-1} = Q_{1,r-1}$. As we shall see, this fact enabled Weir to describe the lower central series of P_r in a particularly nice way :

4.1 THEOREM : (Theorem 3 [18])

The terms in the lower central series of P_r are obtained by removing successive rows from the top of the diagram of P_r.

For the proof we require a few results, which are given here in the more general form for $Q_{n,r}$.

4.2 LEMMA

For m in $\mathbb{Z}^+$,

$$\gamma_m(Q_{n,r}) = [D,{}_{m-1}\langle y_1, \ldots, y_r\rangle][D_1,{}_{m-1}\langle y_2, \ldots, y_r\rangle] \ldots \gamma_m(D_r) .$$

Proof

We proceed by induction on r.

r=1 : $\gamma_m(Q_{n,1}) = \gamma_m(C_{p^n} \operatorname{wr} C_p) = [D,{}_{m-1} Q_{n,1}]\gamma_m(\langle y_1\rangle)$

by Lemma 2.6 i),

$= [D,{}_{m-1}\langle y_1\rangle]\gamma_m(\langle y_1\rangle)$

by Lemma 3.17 iii),

since D is abelian ,

as required.

Now suppose the result is true for $r-1$. Then by associativity of the permutational wreath product,

$$\gamma_m(Q_{n,r}) = \gamma_m(C_{p^n}\ wr^{(p^r)} P_r) \ ,$$

$$= [D,\ _{m-1}Q_{n,r}]\gamma_m(P_r) \quad \text{by Lemma 2.6 i)} \ ,$$

$$= [D,\ _{m-1}P_r][D_1,\ _{m-1}\langle y_2, \dots, y_r\rangle] \times$$

$$[D_2,\ _{m-1}\langle y_3, \dots, y_r\rangle] \dots \gamma_m(\langle y_r\rangle)$$

by Lemma 3.17 iii) , and by hypothesis,

since $P_r \cong Q_{1,r-1}$,

$$= [D,\ _{m-1}\langle y_1, \dots, y_r\rangle][D_1,\ _{m-1}\langle y_2, \dots, y_r\rangle] \dots$$

$$\gamma_m(\langle y_r\rangle) \ ,$$

as required. The result follows by induction. $\square$

4.3 COROLLARY

For m in $\mathbb{Z}^+$, every element in $\gamma_m(Q_{n,r})$ can be expressed uniquely in the form $f d_1 d_2 \dots d_r$, where for $i = 1, \dots, r$, $d_i \in [D_i,\ _{m-1}\langle y_{i+1}, \dots, y_r\rangle]$, and $f \in [D,\ _{m-1}\langle y_1, \dots, y_r\rangle]$.

Proof

By (1) , every element in $Q_{n,r}$ can be expressed in the form $\underline{f}\,\underline{d}_1\underline{d}_2 \dots \underline{d}_r$ where $\underline{f} \in D$, and for $i = 1, \dots, r$, $\underline{d}_i \in D_i$. This expression is unique since $Q_{n,r} = C_{p^n}\ wr^{(p^r)} P_r$ implies $D \cap D_1 \dots D_r = D \cap P_r = \langle 1\rangle$, and $P_{r-i+1} \cong C_p\ wr^{(p^{r-i})} P_{r-i}$ implies $D_i \cap D_{i+1} \dots D_r = \langle 1\rangle$.

Since $D_i \lhd \langle y_i, \dots, y_r\rangle$ for $i = 1, \dots, r$, we have

$[D_i, {}_{m-1}\langle y_{i+1}, \ldots, y_r\rangle] \leq D_i$, and since $D \lhd Q_{n,r}$,

$[D, {}_{m-1}\langle y_1, \ldots, y_r\rangle] \leq D$. The result now follows from Lemma 4.2.

□

4.4 REMARK

Note that for $i = 1, \ldots, r$,

$$[D_i, {}_{m-1}\langle y_{i+1}, \ldots, y_r\rangle] = \gamma_m(Q_{n,r}) \cap D_i = \gamma_m(P_r) \cap D_i ,$$

and

$$[D, {}_{m-1} P_r] = \gamma_m(Q_{n,r}) \cap D .$$

Since $Q_{n,r}$ is nilpotent, we must have by Lemma 4.2 that

$$\gamma_{m+1}(Q_{n,r}) \cap D < \gamma_m(Q_{n,r}) \cap D \quad \text{for } m = 1, \ldots, c(Q_{n,r}) ,$$

and for $i = 1, \ldots, r$, and $m = 1, \ldots, c(\langle y_i, \ldots, y_r\rangle \cong P_{r-i+1})$,

$$\gamma_{m+1}(Q_{n,r}) \cap D_i < \gamma_m(Q_{n,r}) \cap D_i .$$

Thus since $P_{r-i+1} \cong Q_{1,r-i}$ has nilpotency class p^{r-i} by Theorem 3.3 ,

4.5 LEMMA

For $i = 1, \ldots, r$, and $m = 1, \ldots, c(\langle y_i, \ldots, y_r\rangle) = p^{r-i}$,

$$[\gamma_m(P_r) \cap D_i : \gamma_{m+1}(P_r) \cap D_i] = p ,$$

and $\gamma_m(P_r) \cap D_i$ is a group of order $p^{p^{r-i}-m+1}$. □

In the diagram, Figure 3 , $\gamma_m(P_r) \cap D_i$ is represented by the bottom $(p^{r-i} - m+1)$ squares of the column representing D_i .

Proof of Theorem 4.1

The result follows from Lemma 4.5 and the above remark. □

By Lemma 1.5 , $\gamma_m(P_r)$ is generated by simple commutators of length m and $\gamma_{m+1}(P_r)$. Thus since $[\gamma_m(P_r)\cap D_i : \gamma_{m+1}(P_r)\cap D_i]$ is p for $m = 1, \dots , p^{r-i}$, and by Corollary 4.3 every element in $\gamma_m(P_r)$ can be written uniquely in the form $d_1 \dots d_r$ where $d_i \in \gamma_m(P_r)\cap D_i$, it follows that $\gamma_m(P_r)\cap D_i$ is generated by $\gamma_{m+1}(P_r)\cap D_i$ and by any single non-trivial simple commutator of length m in $\gamma_m(P_r)\cap D_i \setminus \gamma_{m+1}(P_r)\cap D_i$. Weir chooses a commutator of the form $[h_1, {}_{m-1} b]$ where $|b| = p^{r-i}$ and $\langle h_1 \rangle$ is the first co-ordinate subgroup of the base group D_i , and so for $i = 1, \dots , r$,

$$D_i = \langle [h_1, {}_j b] : j = 0,1, \dots , p^{r-i} - 1 \rangle . \qquad \dots\dots\dots(2)$$

Ofcourse, this choice reflects that $C_p \text{ wr } C_{p^{r-i}}$ is a subgroup of $\langle y_i, \dots , y_r \rangle$. For the purpose of generalising to $C_{p^n} \text{ wr}^{(p^r)} P_r$ we will choose a different commutator, based on the construction of the $\underline{g}_i$'s of Chapter III . We first establish some notation.

4.6 DEFINITION

Let z_i be the commutator of the same form as the $\underline{g}_i$'s of Definition 3.15 for the group $\langle y_i, \dots , y_r \rangle$, i.e.

$$z_r = y_r ,$$

$$z_{r-1} = [y_{r-1}, {}_{p-1} y_r] ,$$

$$z_{r-2} = [y_{r-2}, {}_{p-1}({}_{p-1} y_r, y_{r-1}), {}_{p-1} y_r] ,$$

and so on. $\square$

4.7 DEFINITION

For $j = 1, \dots, p^{r-i}$, let $z_{i,j}$ be the simple commutator obtained by taking the first j terms in z_i. For example,

$$z_{r-2,2p-1} = [y_{r-2}, {}_{p-1}y_r, y_{r-1}, {}_{p-2}y_r] ,$$

and

$$z_{i,p^{r-i}} = z_i .$$

Let $\underline{Z}$ be the set of $z_{i,j}$'s. □

4.8 LEMMA

For $i = 1, \dots, r$,

i) if $m > p^{r-i}$ then $\gamma_m(P_r) \cap D_i = \langle 1 \rangle$;

and ii) if $1 \leqslant m \leqslant p^{r-i}$ then $\gamma_m(P_r) \cap D_i = \langle z_{i,m}, \gamma_{m+1}(P_r) \cap D_i \rangle$.

Hence $D_i = \langle z_{i,j} : j = 1,2, \dots, p^{r-i} \rangle$.

Proof

Recall that by Theorem 3.3, $c(\langle y_i, \dots, y_r \rangle) = p^{r-i}$, since $\langle y_i, \dots, y_r \rangle \cong P_{r-i+1} \cong Q_{1,r-i}$. The results now follow from the discussion on the preceding page. □

We can think of $z_{i,j}$ as occupying the j th square from the top of the i th column, which represents D_i. ....(3)

A nice result proved by Weir is given as a corollary to Theorem 5 of [18]:

4.9 THEOREM

The upper and lower central series of P_r coincide. □

4.10 REMARK

The cpp-series of P_r is easily determined : by Theorem 4.9 the lower and upper central series of P_r coincide, and so by Remark 1.7 , the cpp-series of P_r coincides with these two series, which accounts for $d(P_r) = p^{r-1}$, as we found in Theorem 3.10 , with $n_1 = n_2 = \dots = n_r = 1$. □

We wish now to construct a table for $C_{p^n} \operatorname{wr}^{(p^r)} P_r$ which also has the property that the lower central series is obtained by removing successive layers, or rows, from the top of the table. Since the order of the base group $D = (C_{p^n})^{(p^r)}$ is p^{np^r} , while from Theorem 3.3 the nilpotency class of $C_{p^n} \operatorname{wr}^{(p^r)} P_r$ is $p^{r-1}\{p + (p-1)(n-1)\}$, which is strictly less than np^r for $n>1$, we see that for $n>1$ some of the factors

$$\{\gamma_m(Q_{n,r}) \cap D\} / \{\gamma_{m+1}(Q_{n,r}) \cap D\}$$

must have order at least p^2. Ofcourse, for $n=1$ we have $C_p \operatorname{wr}^{(p^r)} P_r \cong P_{r+1}$, so our table must coincide with that of Weir, given in Figure 3 , for $n=1$. We will see that all the non-trivial factors $\{\gamma_m(Q_{n,r}) \cap D\} / \{\gamma_{m+1}(Q_{n,r}) \cap D\}$ except for $m=1$, i.e. for $m = 2, \dots , c(Q_{n,r})$, are elementary abelian, and for $m=1$, $D / \{\gamma_2(Q_{n,r}) \cap D\}$ is cyclic of order p^n . In the proof the following commutators play a vital role :

4.11 DEFINITION

For $j = 1, \dots , r$, let $\underline{x}_j$ be the non-trivial commutator in $\langle f_1, y_j, \dots , y_r \rangle \leq Q_{n,r}$ of the same form as the $\underline{g}_i$'s of

Definition 3.15 , where $\langle f_1 \rangle$ is the first co-ordinate subgroup of $D = (C_{p^n})^{(p^r)}$. For example,

$$\underline{x}_r = [f_1, {}_{n(p-1)}y_r] ,$$

$$\underline{x}_{r-1} = [f_1, {}_{n(p-1)}({}_{p-1}y_r, y_{r-1}), {}_{p-1}y_r] ,$$

and $\underline{x}_1 = \underline{g}_r$. ▯

4.12 DEFINITION

For $i = 1, \ldots, r$, and $j = 1, \ldots, p^{r-i}\{p+(p-1)(n-1)\}$, let $x_{i,j}$ be the simple commutator obtained by taking the first j terms of $\underline{x}_i$. For example,

$$x_{r-1,2p-1} = [f_1, {}_{p-1}y_r, y_{r-1}, {}_{p-2}y_r] ,$$

$$x_{r-1,p\{p+(p-1)(n-1)\}} = \underline{x}_{r-1} ,$$

and $x_{1,p^{r-1}\{p+(p-1)(n-1)\}} = \underline{x}_1 = \underline{g}_r$. ▯

4.13 REMARK

i) By construction, $x_{i,j} \in \gamma_j(Q_{n,r}) \setminus \gamma_{j+1}(Q_{n,r})$.

ii) $x_{1,1} = x_{2,1} = \ldots = x_{r,1} = f_1$.

For $k = 0, \ldots, r-1$, and $p^k < j \leqslant p^{k+1}$,

$$x_{1,j} = \ldots = x_{r-k,j} \neq x_{r-k+1,j} .$$ ▯

4.14 DEFINITION

Let $X = \{x_{r,1}, x_{i,j} :$ for $i = 1, \ldots, r$, $j = p^{r-i}+1, p^{r-i}+2, \ldots c(Q_{n,r-i+1})\}$.

Note that X is merely the set of $x_{i,j}$'s of Definition 4.12 with "duplicates" omitted, such that if $x_{i,j} \in X$ then

$x_{i,j} \neq x_{i+1,j}$ for $i = 1, \ldots, r-1$, i.e. in Remark 4.13 ii) we take $x_{r-k,j} \in X$, where $x_{1,j} = \cdots = x_{r-k,j}$. □

4.15 DEFINITION

For $i = 1, \ldots, r$, let X_i be the subset of X consisting of those elements of X of the form $x_{i,j}$, where $j \geq 2$, i.e.

$$X_i = \{ x_{i,j} : j = p^{r-i}+1, p^{r-i}+2, \ldots, c(Q_{n,r-i+1}) \} .$$ □

Note $X = \{ x_{r,1}, X_i : i = 1, \ldots, r \}$.

In order to prove $D / \{ \gamma_2(Q_{n,r}) \cap D \}$ is cyclic of order p^n we require

4.16 LEMMA

Let "Λ, B" be a transitive pair, where Λ is finite. Without loss of generality we may assume $\Lambda = \{ 1, \ldots, k \}$ where $|\Lambda| = k$. Let $W = C_{p^n} \operatorname{wr}^{\Lambda} B$, let $\langle f_1 \rangle = (C_{p^n})_1$, and let f_i be the conjugate of f_1 in $(C_{p^n})_i$ for $i = 2, \ldots, k$. Let $g = \prod_{i=1}^{k} f_i^{t_i}$ where $0 \leq t_i < p^n$ for $i = 1, \ldots, k$. Then

$$g \in \gamma_2(W) \iff \sum_{i=1}^{k} t_i \equiv 0 \mod p^n .$$

Proof

$\Rightarrow$: $g \in \gamma_2(W) \Rightarrow g$ is a finite product of commutators of the form $[f, b]$ where $f \in (C_{p^n})^{\Lambda}$ and $b \in B$, by Lemma 2.6 i) and Lemma 3.17 ii). Now $f = \prod_{i=1}^{k} f_i^{s_i}$ where $0 \leq s_i < p^n$ for

$i = 1, \ldots, k$, and so by Lemma 3.17 ii),

$$[f, b] = \prod_{i=1}^{k} \{ f_i^{-s_i} f_{i.b}^{s_i} \} = \prod_{i=1}^{k} f_i^{-s_i + s_{i.b^{-1}}} ,$$

and $\sum_{i=1}^{k} \{ -s_i + s_{i.b^{-1}} \} = 0$. Hence $\sum_{i=1}^{k} t_i \equiv 0 \mod p^n$.

<u>$\Leftarrow$</u> : $\sum_{i=1}^{k} t_i \equiv 0 \mod p^n$. Note $f_j^{\sum_{i=1}^{k} t_i} = 1$ for all j in $\{1, \ldots, k\}$. Let $b_{ij} \in B$ be such that $i b_{ij} = j$ for $i, j \in \{1, \ldots, k\}$. Let $v_i = \sum_{j=1}^{i} t_j$ for $i = 1, \ldots, k$. Then for $i = 2, \ldots, k$,

$$t_i = -v_{i-1} + v_i ,$$

and

$$g = f_1^{v_1} f_2^{-v_1 + v_2} \ldots f_k^{-v_{k-1} + v_k} f_1^{-v_k} ,$$

$$= [f_1, b_{12}]^{-v_1} [f_2, b_{23}]^{-v_2} \ldots [f_k, b_{k1}]^{-v_k} ,$$

$$\in \gamma_2(W) .$$ □

Lemma 4.16 is a partial generalisation to the transitive permutational wreath product of Theorem 4.1 [13]. In fact, the whole theorem generalises, with only minor modifications to the proof given in [13]. We state the full generalisation below, but omit the proof since we will not use this general version. First we need a definition, which generalises that in [13].

<u>4.17 DEFINITION</u>

Let A be a group and let "Λ, B" be a pair. Let $\underline{\pi}$ be the map from $\mathrm{Dr}\, A^{\Lambda}$ to A such that

$$\underline{\pi}(f) = \prod_{\lambda \in \sigma(f)} f(\lambda) \qquad \forall\ f \in \mathrm{Dr}\, A^{\Lambda} .$$

Note we need to specify an order on the multiplication for $\underline{\pi}$ to be well-defined. However, the following theorem does not require us to specify an order, so we omit it. □

4.18 THEOREM

Let A be a group and let "Λ , B" be a transitive pair such that $|\Lambda| > 1$. Then in the wreath product $A \mathrm{wr}^{\Lambda} B$,

$$[\,\mathrm{Dr}\, A^{\Lambda}, B\,] = \{ f \in \mathrm{Dr}\, A^{\Lambda} : \underline{\pi}(f) \in \gamma_2(A) \} . \qquad □$$

Lemma 4.16 is the case $A = C_{p^n}$, Λ a finite set :

$$g = \prod_{i=1}^{k} f_i^{t_i} \in \gamma_2(W) \iff g \in [\,(C_{p^n})^{\Lambda}, B\,] \quad \text{by Lemma 2.6 i) and Lemma 3.17 ii),}$$

$$\iff \underline{\pi}(g) \in \gamma_2(C_{p^n}) = \langle 1 \rangle \quad \text{by Theorem 4.18,}$$

$$\iff \sum_{i=1}^{k} t_i \equiv 0 \mod p^n .$$

4.19 COROLLARY

Let $\langle f_1 \rangle = (C_{p^n})_1 \leq Q_{n,r}$. Then for $0 < m < p^n$, f_1^{m} belongs to $Q_{n,r} \setminus \gamma_2(Q_{n,r})$.

Proof

$m \not\equiv 0 \mod p^n$, and the result now follows from Lemma 4.16 . □

4.20 LEMMA

Let $\underline{h}$ be a non-trivial simple commutator in $Q_{n,r}$ with first entry f_1 where $\langle f_1 \rangle = (C_{p^n})_1$, and all other entries from the set $\{y_i, \dots, y_r\}$, where $i \in \{1, \dots, r\}$, such that there are $s \neq 0$ entries equal to y_i in $\underline{h}$. Then if the entries after the last y_i in $\underline{h}$ are $b_1, \dots, b_m$, we can re-express $\underline{h}$ as

$$\underline{h} = [f_1, {}_s y_i, b_1, \dots, b_m]^t \qquad \dots\dots.(4)$$

for some t such that $0 < t < p^n$, and thus $\underline{h}$ is a product of conjugates of $[f_1, {}_s y_i]$ with respect to $\langle y_{i+1}, \dots, y_r \rangle$.

<u>Proof</u>

For $j = i+1, \dots, r$, let the orbit of "(p^r) , $\langle y_j \rangle$" containing $\ell \in (p^i) = \sigma(y_i)$ be $\theta_{j\ell}$. Then

$$\theta_{j\ell} = \{\ell, p^{j-1}+\ell, 2p^{j-1}+\ell, \dots, p^j - p^{j-1} + \ell\},$$

and $\theta_{j\ell} \cap \sigma(y_i) = \ell$. Let $c_1, \dots, c_k \in \{y_{i+1}, \dots, y_r\}$. Then for $\ell \in (p^i)$,

$$[f_\ell, c_1, \dots, c_k] = \Pi\{f_\lambda^{v_\lambda} : \lambda = up^i + \ell , \quad 0 \leq u < p^{r-i} , \text{ and } 0 \leq v_\lambda < p^n\} .$$

So $[f_\ell, c_1, \dots, c_k, y_i] = [f_\ell^{v_\ell}, y_i] = [f_\ell, y_i]^{v_\ell}$,

by Lemma 3.17 ii) , since D is abelian.

We now proceed by induction to prove the result (4) .

<u>s=1</u> : This is just the above argument with $\ell=1$.

Suppose the result is true for commutators with $(s-1)$ y_i's . For simplicity, let s_w be such that

$$[f_1, {}_{s-1} y_i] = \prod_{w=0}^{p-1} f_{wp^{i-1}+1}^{s_w} .$$

By the induction hypothesis, if $c_1, \dots, c_k$ are the entries in $\underline{h}$ between the last two y_i's ,

$$\underline{h} = [\, [\, f_1, {}_{s-1}y_i, c_1, \dots, c_k\,]^{t_{s-1}}, y_i, b_1, \dots, b_m\,]$$

where $0 < t_{s-1} < p^n$,

$$= [\, [\, \prod_{w=0}^{p-1} f_{wp^{i-1}+1}^{s_w}, c_1, \dots, c_k\,]^{t_{s-1}}, y_i, b_1, \dots, b_m\,] ,$$

$$= [\, \prod_{w=0}^{p-1} \{ [\, f_{wp^{i-1}+1}, c_1, \dots, c_k, y_i\,]^{s_w} \}, b_1, \dots, b_m\,]^{t_{s-1}}$$

by Lemma 3.17 iii) , since D is abelian ,

$$= [\, \prod_{w=0}^{p-1} \{ [\, f_{wp^{i-1}+1}^{\underline{t}_s}, y_i\,]^{s_w} \}, b_1, \dots, b_m\,]^{t_{s-1}}$$

for some $\underline{t}_s$ such that $0 < \underline{t}_s < p^n$, by the above argument and since $\underline{h} \neq 1$,

$$= [\, [\, \prod_{w=0}^{p-1} f_{wp^{i-1}+1}^{s_w}, y_i\,], b_1, \dots, b_m\,]^{\underline{t}_s t_{s-1}}$$

by Lemma 3.17 iii) ,

$$= [\, f_1, {}_s y_i, b_1, \dots, b_m\,]^{\underline{t}_s t_{s-1}} ,$$

$$= [\, f_1, {}_s y_i, b_1, \dots, b_m\,]^{t_s} , \quad \text{where } t_s = \underline{t}_s t_{s-1} ,$$

and $0 < t_s < p^n$ since $0 < \underline{t}_s, t_{s-1} < p^n$ and f_1 has order p^n.

The result (4) now follows by induction.

The last part follows easily, since the conjugates of $[\, f_1, {}_s y_i\,]$ with respect to $\langle y_{i+1}, \dots, y_r \rangle$ have mutually disjoint supports and thus commute - see pp.17-20 of Chapter I - , and if $b, b' \in \langle y_{i+1}, \dots, y_r \rangle$,

$$[\,[\,f_1,\,{}_s y_i\,]^b,\,b'\,] = [\,f_1,\,{}_s y_i\,]^{-b}\,[\,f_1,\,{}_s y_i\,]^{b\,b'}\,,$$

which is a product of conjugates of $[\,f_1,\,{}_s y_i\,]$ with respect to $\langle y_{i+1}, \ldots , y_r \rangle$: a trivial induction argument on m yields the result. $\square$

4.21 LEMMA

$\langle X \rangle = \langle f_1 \rangle \times \langle X_1 \rangle \times \langle X_2 \rangle \times \ldots \times \langle X_r \rangle$, where definitions are given in Definitions 4.12, 4.14, and 4.15 .

Proof

By definition, $\langle X_i \ : \ i = 1, \ldots , r \rangle \in \gamma_2(Q_{n,r})$, and by Corollary 4.19 , for $0 < t < p^n$, $f_1^{\,t} \in Q_{n,r} \setminus \gamma_2(Q_{n,r})$. Hence $\langle f_1 \rangle \cap \langle X_i \ : \ i = 1, \ldots , r \rangle = \langle 1 \rangle$, and so we need to consider $\langle X_i \ : \ i = 1, \ldots , r \rangle$. We wish to show that if $k \in \{ 1, \ldots , r \}$, then $\langle X_k \rangle \cap \langle X_j \ : \ j \in \{ 1, \ldots , r \} \setminus \{ k \} \rangle = \langle 1 \rangle$, since this implies the result as D is abelian. The argument is by comparison of supports.

For $i \in \{ 1, \ldots , r \}$, if $x_{i,j} \in X_i$, then $x_{i,j}$ contains at least one entry equal to y_i , by construction. Hence, if $h_i \in \langle X_i \rangle$, then by Lemma 4.20, h_i is a product of conjugates of $[\,f_1,\,{}_s y_i\,]$ with respect to $\langle y_{i+1}, \ldots , y_r \rangle$ for various s such that $0 < s \leqslant n(p-1)$.

Since for $k > i$, y_k "shifts up" points in (p^r) by $p^{k-1} \geqslant p^i$,

$$\sigma(h_i) \subseteq \{ u p^i + 1, u p^i + p^{i-1} + 1, \ldots , (u+1)p^i - p^{i-1} + 1 : u = 0, \ldots , p^{r-i} - 1 \} . \qquad \ldots\ldots.(5)$$

We show that if $h_i \neq 1$, then

$$\sigma(h_i) \nsubseteq \{ u p^i + 1 : u = 0, \ldots , p^{r-i} - 1 \} ,$$

i.e. there exists λ in $\sigma(h_i)$ of the form

$$\lambda = up^i + v p^{i-1} + 1 \qquad \text{where } u \in \{ 0, \ldots , p^{r-i} - 1 \} \text{ and } v \in \{ 1, \ldots , p-1 \} . \qquad \ldots\ldots.(6)$$

Since $1 \neq f_1^t \in Q_{n,r} \setminus \gamma_2(Q_{n,r})$, by Corollary 4.19 , and $\langle X_i \rangle \leq \gamma_2(Q_{n,r})$,

$$1 \neq \prod_{s=1}^{n(p-1)} [f_1, {}_s y_i]^{t_s} \neq f_1^t \qquad \text{for any choice of } t \text{ such that } 0 < t < p^n \text{, and any choice of } t_s \text{ for } s = 1, \ldots , n(p-1) \text{, not all } 0 .$$

Thus there exists μ in (p^r) such that

$$1 \neq \mu \in \sigma(\prod_{s=1}^{n(p-1)} [f_1, {}_s y_i]^{t_s}) ,$$

$$\subseteq \{ 1, p^{i-1} + 1, \ldots , p^i - p^{i-1} + 1 \} ,$$

the orbit of y_i containing 1 . For $u = 0, \ldots , p^{r-i} - 1$, let $b_u \in \langle y_{i+1}, \ldots , y_r \rangle$ be such that $1.b_u = u p^i + 1$. Then

$$\mu b_u = up^i + \mu \in \sigma(\{ \prod_{s=1}^{n(p-1)} [f_1, {}_s y_i]^{t_s} \}^{b_u}) \setminus \{ u p^i + 1 \} ,$$

$$\subseteq \{ up^i + p^{i-1} + 1, up^i + 2p^{i-1} + 1, \ldots , (u+1)p^i - p^{i-1} + 1 \} . \qquad \ldots\ldots.(7)$$

We may re-express $h_i \in \langle X_i \rangle$ in the form

$$h_i = \Pi\{ h_{iu} \ : \ u = 0, \dots , p^{r-i} - 1 \}$$

where h_{iu} is of the form $\{ \prod_{s=1}^{n(p-1)} [f_1, {}_s y_i]^{t_s} \}^{b_u}$, and since the sets $\sigma(h_{iu}) \subseteq \{ u p^i + 1, \dots , (u+1)p^i - p^{i-1} + 1 \}$ are mutually disjoint for distinct u, $\sigma(h_{iu}) \subseteq \sigma(h_i)$, for $u = 0, \dots$, $p^{r-i} - 1$. Thus if $h_i \neq 1$, by (7) there exists u in $\{ 0, \dots , p^{r-i} - 1 \}$ and v in $\{ 1, \dots , p-1 \}$ such that

$$\lambda = u p^i + v p^{i-1} + 1 \in \sigma(h_i) ,$$

and we have proved (6).

We now show if $1 \neq h_k \in \langle X_k \rangle$ and $h \in \langle X_j \ : \ j \in \{ 1, \dots , r \} \setminus \{k\} \rangle$, then $\sigma(h_k) \neq \sigma(h)$, and so $h_k \neq h$, i.e.

$$\langle X_k \rangle \cap \langle X_j \ : \ j \in \{ 1, \dots , r \} \setminus \{k\} \rangle = \langle 1 \rangle .$$

If $h \in \langle X_j \ : \ j \in \{ 1, \dots , r \} \setminus \{k\} \rangle$, then since D is abelian, $h = \Pi\{ h_j \ : \ h_j \in \langle X_j \rangle , \ j \in \{ 1, \dots , r \} \setminus \{k\} \}$: note we are not assuming uniqueness of such an expression. Then

$$\begin{aligned} \sigma(h) &\subseteq \cup\{ \sigma(h_j) \ : \ j \in \{ 1, \dots , r \} \setminus \{k\} \} , \\ &\subseteq \{ u_j p^j + v_j p^{j-1} + 1 : \ j \in \{ 1, \dots , r \} \setminus \{k\} , \\ &\qquad\qquad u_j \in \{ 0, \dots , p^{r-j} - 1 \} , \text{ and} \\ &\qquad\qquad v_j \in \{ 0, \dots , p-1 \} \} , \text{ by } (5) . \end{aligned}$$

Let $\lambda = u p^k + v p^{k-1} + 1 \in \sigma(h_k)$ where $u \in \{ 0, \dots , p^{r-k} - 1 \}$ and $v \in \{ 1, \dots , p-1 \}$: such a λ exists by (6). Then

$$\begin{aligned} &\lambda \in \sigma(h) \\ &\Rightarrow \lambda (= u p^k + v p^{k-1} + 1 = (u p + v) p^{k-1} + 1 ,) \\ &\qquad \in \{ u_j p^j + 1 : \ j \in \{ 1, \dots , k-1 \} , \ u_j \in \{ 0, \dots , p^{r-j} - 1 \} \} \end{aligned}$$

since v is in $\{1, \dots, p-1\}$. If $k = 1$ then this shows λ is not in $\sigma(h)$, and so $\sigma(h_k) \neq \sigma(h)$. So suppose now $k \neq 1$, and $\lambda = u_j' p^j + 1 \in \sigma(h_j \neq 1) \subseteq \sigma(h)$. Then again by (6), there exists μ in $\sigma(h_j) \subseteq \sigma(h)$ such that $\mu = u_j' p^j + v_j' p^{j-1} + 1$ where u_j' is as above and v_j' is in $\{1, \dots, p-1\}$. But since $v_j' \neq 0$,

$$\begin{aligned} \mu(= u_j' p^j + v_j' p^{j-1} + 1\ ,) \\ \notin \{ u_k p^k + v_k p^{k-1} + 1 : \ & u_k \in \{0, \dots, p^{r-k} - 1\}, \\ & v_k \in \{0, \dots, p-1\}\}, \end{aligned}$$

and so by (5), $\mu \notin \sigma(h_k)$. Hence $\sigma(h_k) \neq \sigma(h)$, and we obtain the result. $\square$

<u>4.22 COROLLARY</u>

No proper subset of X is a basis for $D = (C_{p^n})^{(p^r)}$.

<u>Proof</u>

By Lemma 4.21, it remains to show that if $x_{i,j} \in X_i$ for some $i \in \{1, \dots, r\}$, then $x_{i,j}$ is not a product of elements from $X_i \setminus \{x_{i,j}\}$. This is immediate since the elements in X_i are all by definition of different nilpotency weights, as can be seen from Remark 4.13 i), and the result follows. $\square$

<u>4.23 THEOREM</u>

The set X is a basis for the base group $D = (C_{p^n})^{(p^r)}$ of $Q_{n,r}$. In particular,

i) for $k = 1, \dots, c(Q_{n,r})$, $\gamma_k(Q_{n,r}) \cap D$ is generated modulo $\gamma_{k+1}(Q_{n,r}) \cap D$ by the set $\{x_{i,k} : x_{i,k} \in X\}$;

and ii) $D/\{\gamma_2(Q_{n,r}) \cap D\}$ is cyclic of order p^n, and for $k = 2, \ldots, c(Q_{n,r})$, $\{\gamma_k(Q_{n,r}) \cap D\} / \{\gamma_{k+1}(Q_{n,r}) \cap D\}$ is elementary abelian.

The proof follows shortly. From Lemma 4.21 and Theorem 4.23 we obtain the following diagram for D :

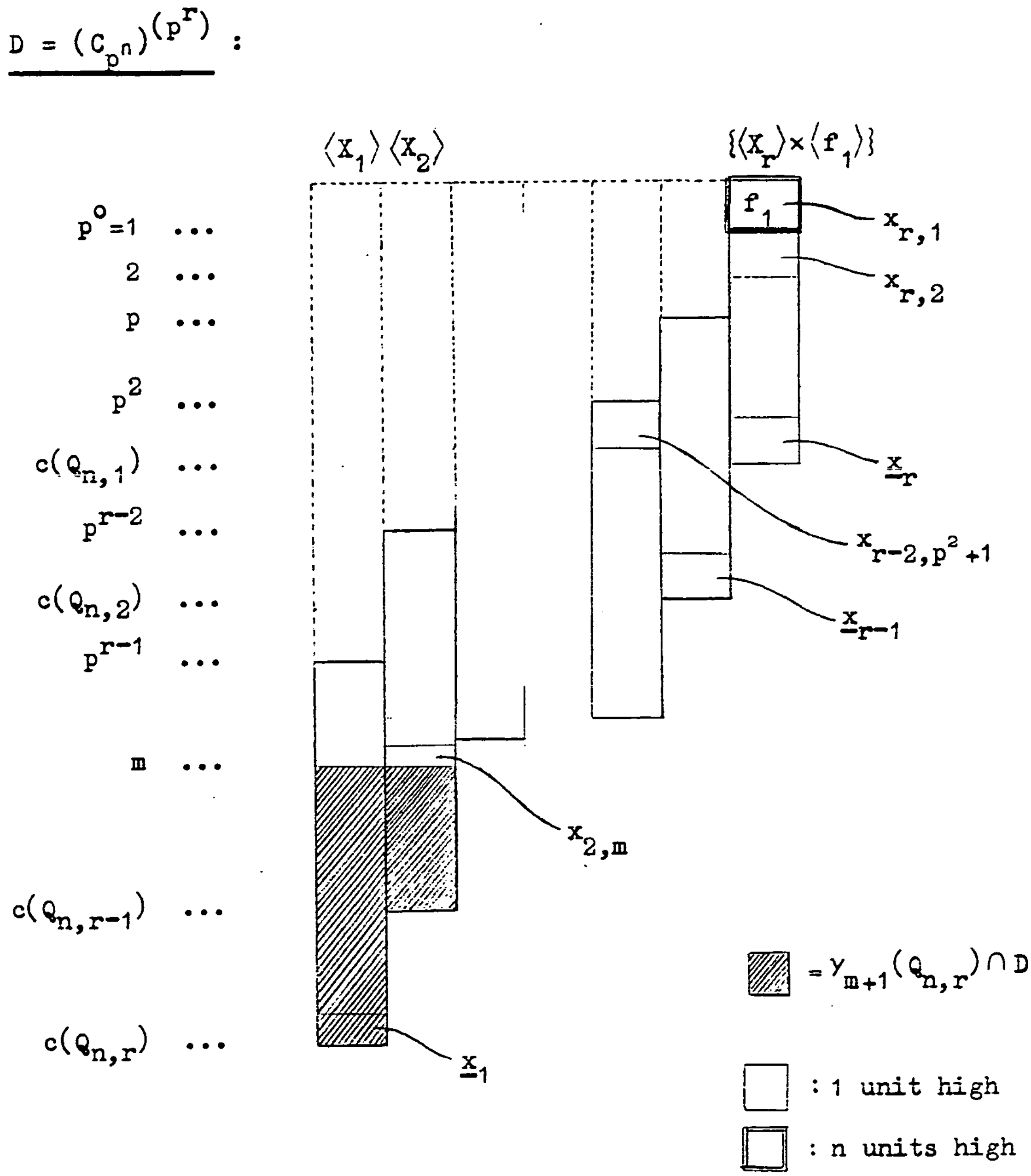


Fig. 4

In general, for $i = 1, \dots, r-1$, the part of the ith column in solid line represents $\langle X_i \rangle$, and the rth column represents $\langle X_r \rangle \times \langle f_1 \rangle$. Note $f_1 = x_{r,1}$, contained in the first box of the rth column. This box, the $r,1$th box, is n units high to indicate that $(x_{r,1} = f_1)^{p^n}$ belongs to $\gamma_{1+1}(Q_{n,r}) \cap D$, but $x_{r,1}^{p^{n-1}}$ does not belong to $\gamma_2(Q_{n,r}) \cap D$. For $\ell \geq 2$, $k = 1, \dots, r$, the element $x_{k,\ell}$ of X is contained in the k,ℓth box, which is 1 unit high to indicate $x_{k,\ell}^{p}$ is in $\gamma_{\ell+1}(Q_{n,r}) \cap D$, but $x_{k,\ell}$ is not in $\gamma_{\ell+1}(Q_{n,r}) \cap D$. The subgroups $\gamma_m(Q_{n,r}) \cap D$ are obtained by removing successive layers from the top of the diagram.

Note that for $n=1$, we have $Q_{n,r} = Q_{1,r} \cong P_{r+1}$, for which we know the factors in the base group are cyclic of order p. By Definitions 4.11 and 4.12, for $i = 1, \dots, r$, $\underline{x}_i$ is just $x_{i,c(Q_{1,r-i+1})}$, and by Theorem 3.3, $c(Q_{1,r-i+1} \cong P_{r-i+2})$ is p^{r-i+1}. Thus we obtain the diagram overleaf for P_{r+1} from Figure 4, for $n=1$. The diagram indicates all factors in the base group are indeed of order p, as required.

<u>Proof of Theorem 4.23</u>

By Corollary 4.19, for $0 < t < p^n$, $f_1^{\,t} \notin \gamma_2(Q_{n,r})$. Hence $[D : \{\gamma_2(Q_{n,r}) \cap D\}] \geq p^n$.

For $\ell = 2, \dots, c(Q_{n,r})$, let v_ℓ be the number of $x_{i,\ell}$'s in X. Then since no $x_{i,\ell}$ in X is a product of elements from $X \setminus \{x_{i,\ell}\}$, by Corollary 4.22,

$\underline{D = (C_p)^{(p^r)}}$:

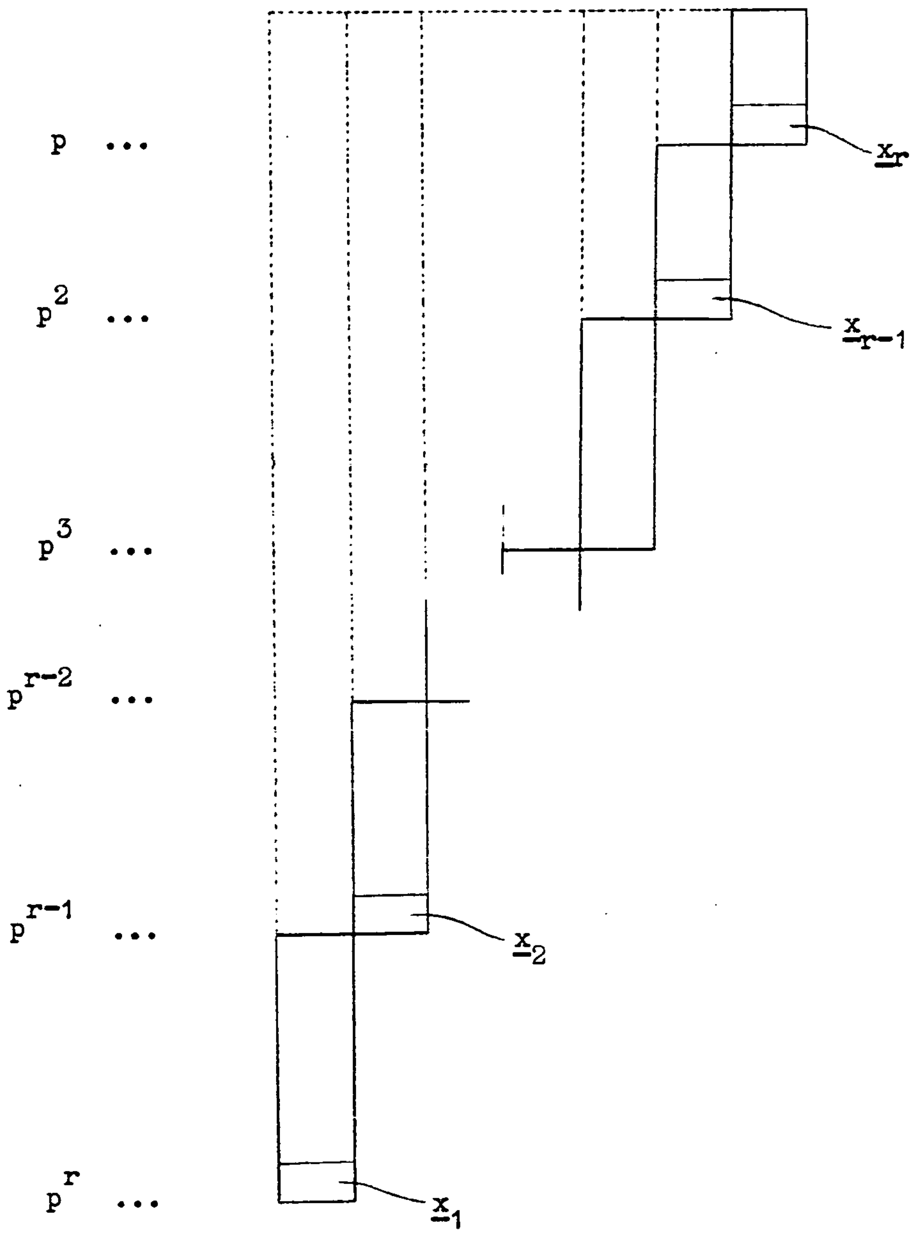


Fig. 5

$$[\{\gamma_\ell(Q_{n,r}) \cap D\} : \{\gamma_{\ell+1}(Q_{n,r}) \cap D\}] \geq p^{v_\ell} .$$

Now $p^{np^r} = |D| = \prod_{i=1}^{c(Q_{n,r})} [\{\gamma_i(Q_{n,r}) \cap D\} : \{\gamma_{i+1}(Q_{n,r}) \cap D\}]$, which is greater than or equal to $p^{n+\sum_{\ell=2}^{c(Q_{n,r})} v_\ell}$ . ...(8)

But by definition of the v_ℓ's ,

$$|X| = 1 + \sum_{\ell=2}^{c(Q_{n,r})} v_\ell ,$$

and by definition of X ,

$$\begin{aligned} |X| &= 1 + \sum_{i=1}^{r} \{ c(Q_{n,r-i+1}) - p^{r-i} \} , \\ &= 1 + \sum_{i=1}^{r} \{ p^{r-i}\{ p + (p-1)(n-1)\} - p^{r-i} \} \quad \text{by Theorem 3.3} , \\ &= 1 + n(p-1) \sum_{i=1}^{r} p^{r-i} , \\ &= 1 + n(p^r - 1) , \\ &= np^r - (n-1) . \end{aligned}$$

Hence $np^r = n + \sum_{\ell=2}^{c(Q_{n,r})} v_\ell$, and so we have equality in (8) . It follows immediately that $[D : \{ \gamma_2(Q_{n,r}) \cap D \}] = p^n$, and for $\ell = 2, \dots , c(Q_{n,r})$, $[\{ \gamma_\ell(Q_{n,r}) \cap D \} : \{ \gamma_{\ell+1}(Q_{n,r}) \cap D \}] = p^{v_\ell}$, i.e. $\gamma_\ell(Q_{n,r}) \cap D$ is generated by $\gamma_{\ell+1}(Q_{n,r}) \cap D$ and the set $\{ x_{i,\ell} : x_{i,\ell} \in X \}$, and for $x_{i,\ell}$ in X , $x_{i,\ell}^p$ belongs to $\gamma_{\ell+1}(Q_{n,r}) \cap D$. The results of the theorem now follow. □

4.24 COROLLARY

For $i = 2, \dots , c(Q_{n,r})$, $\gamma_i(Q_{n,r}) / \gamma_{i+1}(Q_{n,r})$ is elementary abelian.

Proof

By Theorem 4.1 , for $i = 2, \dots , p^{r-1} = c(P_r)$, $\gamma_i(P_r) / \gamma_{i+1}(P_r)$

is elementary abelian. The corollary now follows from Lemma 2.6 i) and Theorem 4.23 , since $Q_{n,r} = C_{p^n} \operatorname{wr}^{(p^r)} P_r$. □

As an immediate corollary to Theorem 4.23 and Lemma 4.21 we have

4.25 COROLLARY

Every element in $D \leqslant Q_{n,r}$ is uniquely expressible in the form

$\Pi\{ x_{i,j}^{t_{i,j}} : x_{i,j} \in X , \ 0 \leqslant t_{r,1} < p^n$, and for $i,j \neq r,1$,
$0 \leqslant t_{i,j} < p \}$. □

To give an idea of the orders of the factors $\{ \gamma_i(Q_{n,r}) \cap D \} / \{ \gamma_{i+1}(Q_{n,r}) \cap D \}$ for $i = 1, \ldots , c(Q_{n,r})$, we provide the two following diagrams, which can easily be deduced from Figure 4. Let k in $\mathbb{Z}^+$ be such that $p^k \leqslant p + (p-1)(n-1) < p^{k+1}$, i.e. $p^k \leqslant c(Q_{n,1}) < p^{k+1}$. Recall that $C_{p^n} \operatorname{wr} C_p = Q_{n,1}$. Note that $n > k$ for $n \geqslant 2$:

$$\begin{aligned} & p + (p-1)(n-1) \geqslant p^k \\ \Rightarrow\ & np > p^k , \\ \Rightarrow\ & n > p^{k-1} , \end{aligned}$$

and $p^{k-1} \geqslant k$ for k in $\mathbb{Z}^+$, as an easy induction shows.

In each of the following diagrams, everything to the right of the i th column, i.e. the $(i+1)$ th column to the $c(Q_{n,r})$ th column, represents $\gamma_{i+1}(Q_{n,r}) \cap D$, for $i = 1, \ldots , c(Q_{n,r}) - 1$, and if the height of the i th column is m , for i in $\{ 1, \ldots , c(Q_{n,r}) \}$, then $\{ \gamma_i(Q_{n,r}) \cap D \} / \{ \gamma_{i+1}(Q_{n,r}) \cap D \}$ has order p^m .

$\underline{D}$: $k < r$

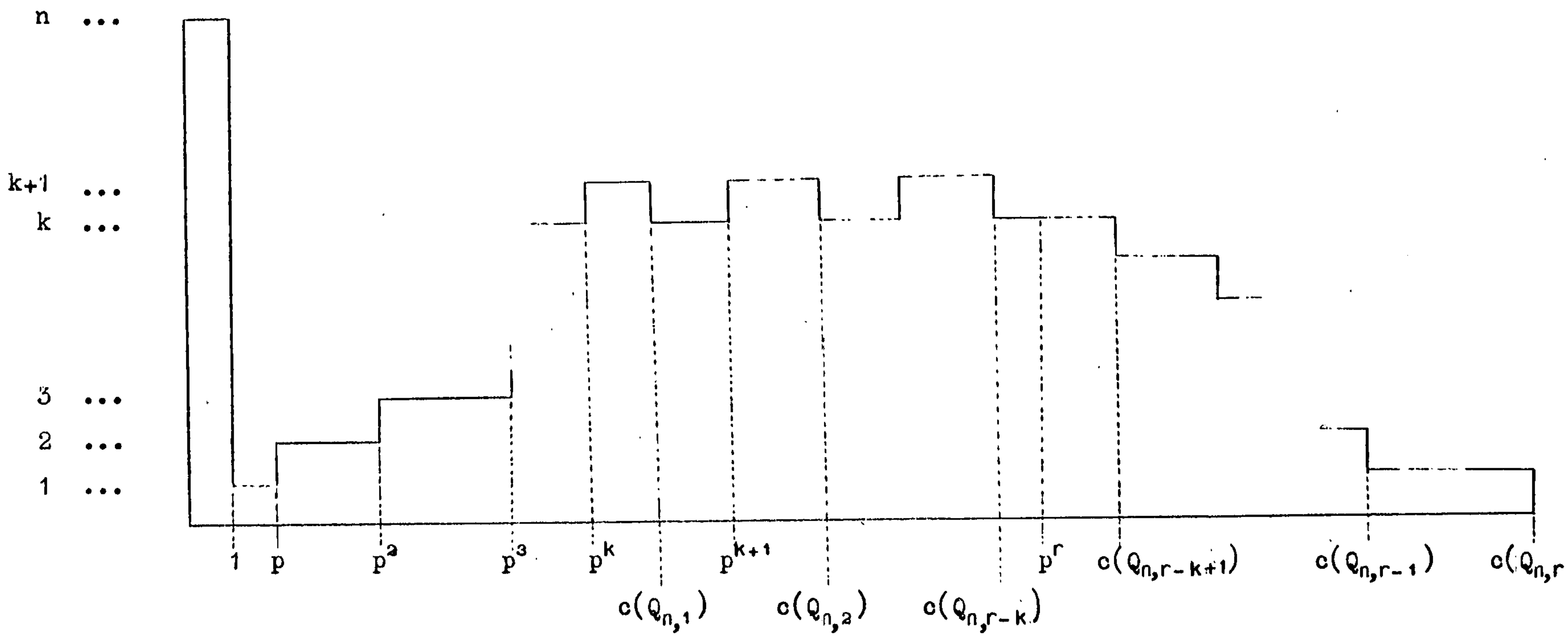


Fig. 6

$\underline{D}$: $k \geq r$

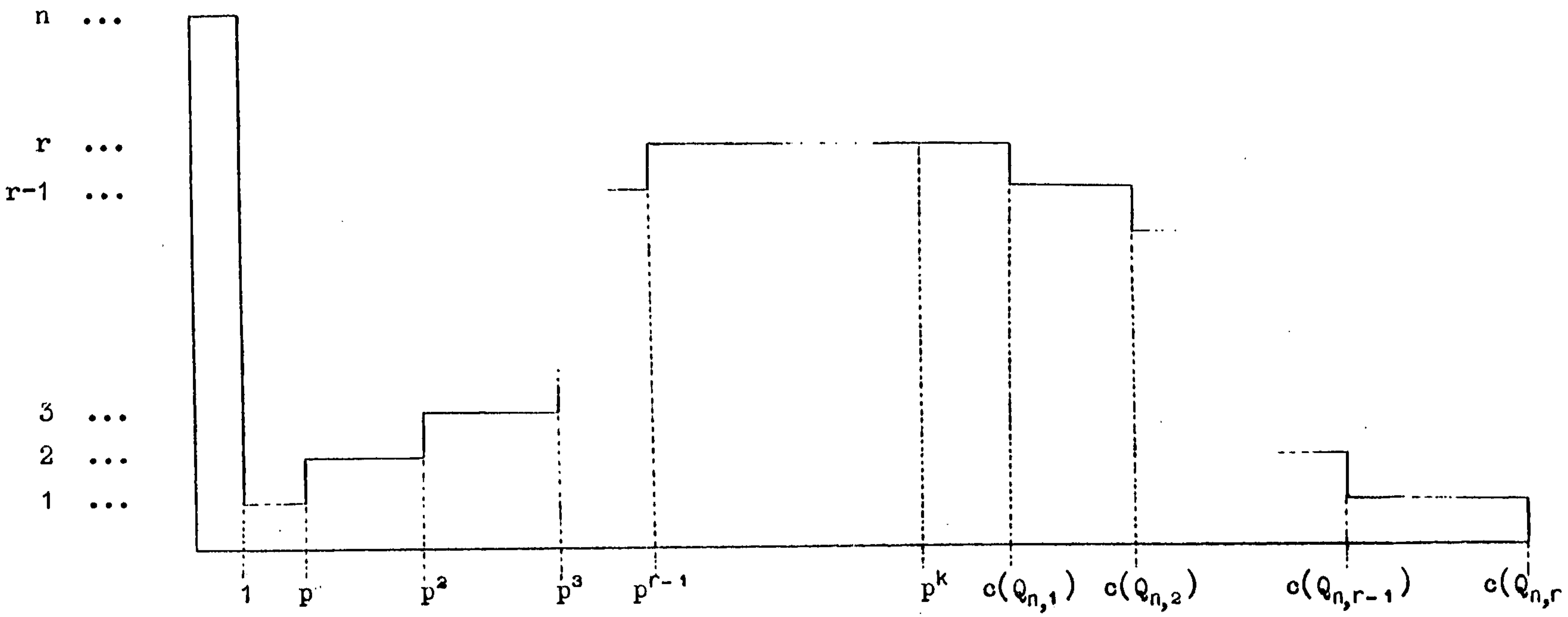


Fig. 7

Note if $n=1$ then $k=1 \leq r$, and "$p^k = p+(p-1)(n-1)$", so no column is more than $1=n$ high, as required for $Q_{1,r} \cong P_{r+1}$.

Combining either Figure 6 or Figure 7, according to whether $k<r$ or $k \geq r$, with Figure 3, we obtain a diagram for $Q_{n,r}$ which generalises that of P_r given in Figure 3. This is given as Figure 8 overleaf : note that the exact shape of the part representing D is that of the appropriate figure above, Figure 6 or Figure 7. As in the diagram for P_r, the lower central series of $Q_{n,r}$ is obtained by removing successive layers from the North. However, although the orders of the lower central factors are represented more clearly in Figure 8 than in a combination of Figure 3, (3), and Figure 4, this latter combination exhibits the internal structure of $Q_{n,r}$ rather better.

As pointed out in Remark 4.9, the cpp-, upper central and lower central series of P_r coincide. No such result holds for $Q_{n,r}$ for $n>1$: by Corollary 3.11, and Theorem 3.3, for $n>1$,

$$d(Q_{n,r}) = p^{n+r-1} \neq c(Q_{n,r}) = p^{r-1}\{p+(p-1)(n-1)\},$$

and so we can expect different structures for the three series.

4.26 REMARK

Note that the relationship between the U.C.S. and L.C.S. of $Q_{n,r}$ depends considerably on the relationship between n and r. Recall that by Corollary 2.33, if $\langle f_1 \rangle = (C_{p^n})_1$ as usual,

$$Z(Q_{n,r}) = \left\langle \prod_{i=1}^{p^r} f_i \;:\; f_i \text{ is the conjugate of } f_1 \text{ in } (C_{p^n})_i \right\rangle.$$

$\underline{Q_{n,r}}$:

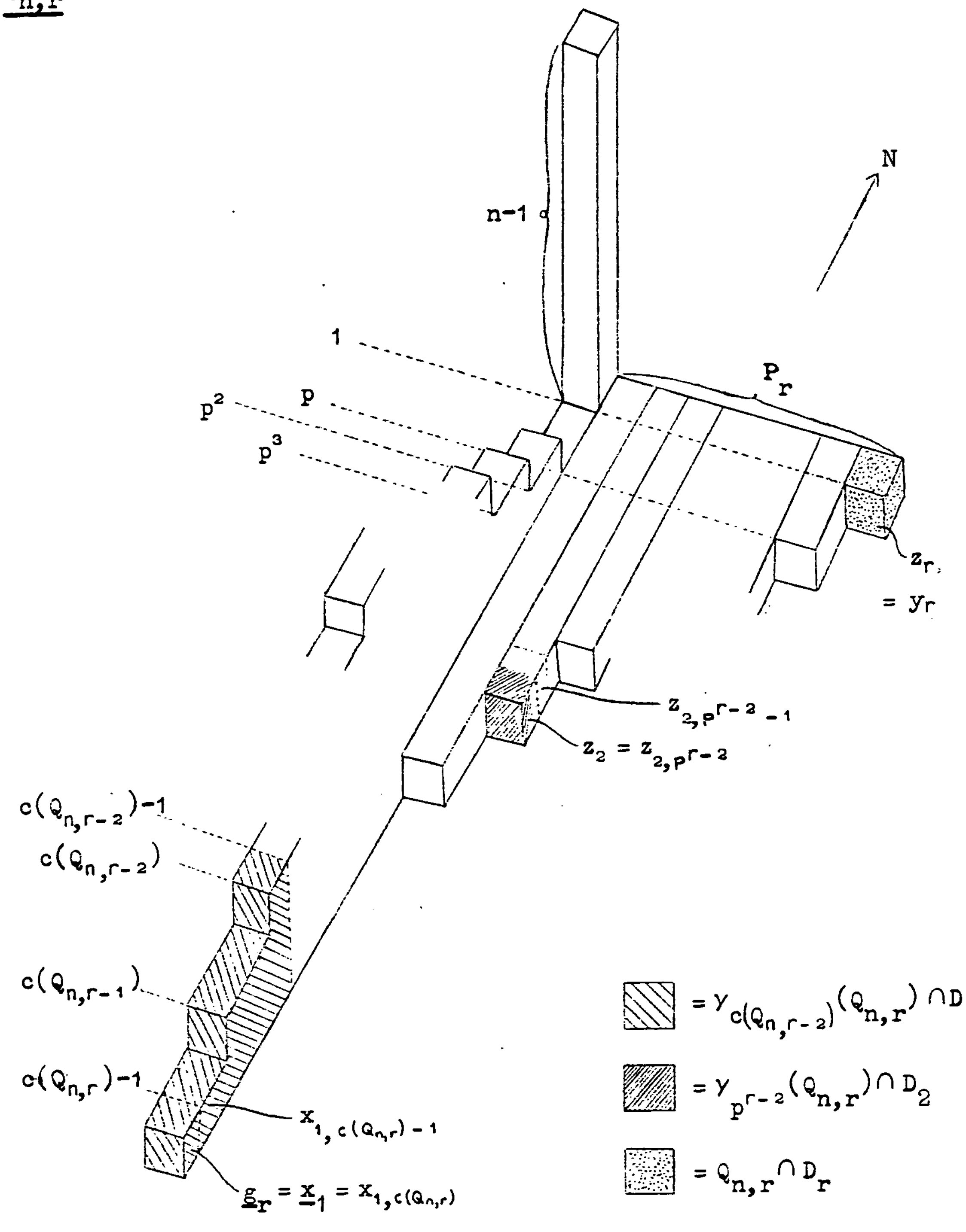


Fig. 8

If $n \leq r$ then $\sum_{i=1}^{p^r} 1 = p^r$ is divisible by p^n, and so by Lemma 4.16, $Z(Q_{n,r}) \leq \gamma_2(Q_{n,r})$. However, if $r < n$ then $p^r \not\equiv 0 \mod p^n$, and so by Lemma 4.16, $Z(Q_{n,r}) \not\leq \gamma_2(Q_{n,r})$. $\square$

We now determine the cpp-series of $Q_{n,r}$ in relation to the lower central series of $Q_{n,r}$. First we prove a theorem for the cpp-series of a permutational wreath product with abelian bottom group analogous to Lemma 2.6 i). Note we give the theorem only for cpp-nilpotent wreath products.

<u>4.27 THEOREM</u>

Let A be an abelian group and let "Λ, B" be a pair such that $W = A \text{ wr}^{\Lambda} B$ is cpp-nilpotent for the prime p. Then for i in $\mathbb{Z}^+$,

$$\pi_i(W) = \langle [A^{\Lambda},{}_{u-1} W]^{p^h},\ \pi_i(B) \ :\ up^h \geq i \rangle .$$

For the proof we require two lemmas.

<u>4.28 LEMMA</u>

Let "Λ, B" be a pair, and let A be an abelian group. Let $f \in A^{\Lambda} \leq A \text{ wr}^{\Lambda} B$, and let $b \in B$. Then for m in $\mathbb{N} \setminus \{0,1\}$,

$$[f, b^m] = [f, b]^{\binom{m}{1}} [f,{}_2 b]^{\binom{m}{2}} \ldots [f,{}_m b]^{\binom{m}{m}} .$$

<u>Note</u> For $m = p$ this is related to Lemma 3.6 of [17].

<u>Proof</u>

We proceed by induction on m.

$\underline{m=2}$: $[f, b^2] = [f, b][f, b][f, b, b]$ by Lemma 1.1 ii) ,

$$= [f, b]^{\binom{2}{1}} [f, {}_2 b]^{\binom{2}{2}} \text{ as required.}$$

Now suppose the result is true for m . Then

$$[f, b^{m+1}] = [f, b][f, b^m][f, b^m, b] \quad \text{by Lemma 1.1 ii),}$$

$$= [f, b] \prod_{j=1}^{m} [f, {}_j b]^{\binom{m}{j}} \left[\prod_{j=1}^{m} [f, {}_j b]^{\binom{m}{j}}, b \right]$$

by the induction hypothesis,

$$= [f, b] \prod_{j=1}^{m} [f, {}_j b]^{\binom{m}{j}} \prod_{j=1}^{m} [f, {}_{j+1} b]^{\binom{m}{j}}$$

by Lemma 3.17 ii) , since A^{Λ} is abelian,

$$= [f, b]^{1+\binom{m}{1}} [f, {}_2 b]^{\binom{m}{2}+\binom{m}{1}} \ldots [f, {}_m b]^{\binom{m}{m}+\binom{m}{m-1}} \times [f, {}_{m+1} b]$$

since A^{Λ} is abelian,

$$= [f, b]^{\binom{m+1}{1}} [f, {}_2 b]^{\binom{m+1}{2}} \ldots [f, {}_{m+1} b]^{\binom{m+1}{m+1}} ,$$

as required for the induction. □

<u>4.29 LEMMA</u> (Lemma 2.1 [11])

$$\sum_{i=1}^{n} \binom{i+r-1}{r} = \binom{n+r}{r+1} .$$ □

<u>Proof of Theorem 4.27</u>

By Lemma 1.2 , and since W is cpp-nilpotent,

$$\pi_i(W) = \left\langle g^{p^t} : g \in \gamma_v(W) \setminus \gamma_{v+1}(W) , \ v p^t \geq i \right\rangle .$$

By Lemma 2.6 i) , if $g \in \gamma_v(W)$ then $g = f b$ where

$f \in [A^{\Lambda}, {}_{v-1} W] \leqslant A^{\Lambda}$, which is abelian, and $b \in \gamma_v(B)$. Hence

$$g^{p^t} = (fb)^{p^t} ,$$

$$= f^{1+b+b^2+\dots+b^{p^t-1}} b^{p^t} .$$

Clearly $b^{p^t} \in \pi_i(B)$ since $b \in \gamma_v(B)$ and $vp^t \geqslant i$. Since

$$f^{b^i} = f[f, b^i] ,$$

$$f^{1+b+\dots b^{p^t-1}} = f\, f[f, b]\, f[f, b^2] \dots f[f, b^{p^t-1}] ,$$

$$= f^{p^t} [f, b]^{\binom{1}{1}+\binom{2}{1}+\dots+\binom{p^t-1}{1}} \times$$

$$[f, {}_2 b]^{\binom{2}{2}+\binom{3}{2}+\dots+\binom{p^t-1}{2}} \dots$$

$$[f, {}_{p^t-1} b]^{\binom{p^t-1}{p^t-1}} \quad \text{by Lemma 4.28}$$

and since A^{Λ} is abelian,

$$= f^{p^t} [f, b]^{\binom{p^t}{2}} [f, {}_2 b]^{\binom{p^t}{3}} \dots [f, {}_{p^t-1} b]^{\binom{p^t}{p^t}}$$

by Lemma 4.29 .

By choice of $g \in \gamma_v(W)$, f and b have nilpotency weight at least v , and so $[f, {}_i b] \in \gamma_{v(i+1)}(W) \cap A^{\Lambda} = [A^{\Lambda}, {}_{v(i+1)-1} W]$ by Lemma 2.6 i) . Then if p^k is the highest power of p dividing $\binom{p^t}{i+1}$, $[f, {}_i b]^{\binom{p^t}{i+1}} \in [A^{\Lambda}, {}_{v(i+1)-1} W]^{p^k} \leqslant \pi_{v(i+1)p^k}(W)$. But if $i+1$ is not a pth power, then p^t divides $\binom{p^t}{i+1}$ and if $i+1$ is a pth power then p^t divides $(i+1)\binom{p^t}{i+1}$, and so $v(i+1)p^k \geqslant vp^t \geqslant i$. Thus

$$g^{p^t} \in \left\langle [A^{\Lambda}, {}_{u-1} W]^{p^h} , \pi_i(B) : up^h \geqslant i \right\rangle .$$

The reverse inclusion is clear. □

4.30 COROLLARY

$$\pi_i(Q_{n,r} = C_{p^n}\ wr^{(p^r)} P_r) = \langle [D,\ _{u-1} Q_{n,r}]^{p^h} ,\ \pi_i(P_r) : up^h \geq i \rangle .$$

□

We know the structure of the cpp-series of P_r , which is just the L.C.S. of P_r , as we saw in Remark 4.10 . Hence to determine the cpp-series of $Q_{n,r}$ we need consider only

$$\pi_i(Q_{n,r}) \cap D = \langle [D,\ _{u-1} Q_{n,r}]^{p^h} : up^h \geq i \rangle$$

for $i = 1, \ldots , d(Q_{n,r})$. By Corollary 3.11 , $d(Q_{n,r}) = p^{n+r-1}$. In fact, we will see it is sufficient to calculate the cpp-weights of the elements in the set X , for :

4.31 LEMMA

Let $i \in \{1, \ldots , d(Q_{n,r})\}$. Then $\pi_i(Q_{n,r}) \cap D$ is generated modulo $\pi_{i+1}(Q_{n,r}) \cap D$ by those $x_{k,\ell}$ in X of cpp-weight i , and by $f_1^{p^m}$ if $i = p^m$ for some m such that $0 \leq m \leq n-1$.

For the proof we require a few results.

4.32 LEMMA

Let i belong to $\{1, \ldots , r\}$. Then the elements in the set X_i have distinct cpp-weights.

Proof

Suppose $j < j'$ and $x_{i,j}, x_{i,j'}$ are in X_i . Then by definition, there exist $b_1, \ldots , b_{j'-j} \in \langle y_i, \ldots , y_r \rangle$ such that

$$x_{i,j'} = [x_{i,j}, b_1, \ldots , b_{j'-j}] .$$

Thus if $x_{i,j}$ has cpp-weight w, then $x_{i,j'}$ has cpp-weight at least $w+j'-j > w$, and the result follows. $\square$

4.33 LEMMA

Let $\{i_1, i_2, \ldots, i_s\} \subseteq \{1, \ldots, r\}$, where $i_1, \ldots, i_s$ are distinct, and suppose $x_{i_1,j_1}, x_{i_2,j_2}, \ldots, x_{i_s,j_s}$ in $X \setminus \{f_1 = x_{r,1}\}$ all have cpp-weight w. If

$$h = \prod_{k=1}^{s} x_{i_k,j_k}^{t_{i_k,j_k}}$$

where $0 < t_{i,j} \leq p-1$ for $i,j = i_1,j_1, \ldots, i_s,j_s$, then h has cpp-weight w. Furthermore, if $i = p^m$, where $0 \leq m \leq n-1$, $f_1^{tp^m} h$ has cpp-weight p^m for $0 < t < p$.

Proof

Certainly h has cpp-weight at least w. Suppose for contradiction that h has cpp-weight strictly greater than w. Then by Lemma 1.2 and Corollary 4.30, h is a product of elements in D of the form g^{p^v} where $g \in \gamma_u(Q_{n,r}) \cap D$ and $up^v > w$. Express each g in the form given by Corollary 4.25. Then $h = h_1 \ldots h_r$, where for $\ell = 1, \ldots, r$,

$$h_\ell = \prod_j \{ x_{\ell,j}^{a_{0,j}+a_{1,j}p+\cdots+a_{n-1,j}p^{n-1}} : \text{for } q = 0, \ldots, n-1,\ 0 \leq a_{q,j} < p, \text{ and if there exists a smallest integer } q' \text{ for which } a_{q',j} \neq 0, \text{ i.e. the power of } x_{\ell,j} \text{ in } h_\ell \text{ is non-trivial, then } jp^{q'} > w\},$$

which implies $h_\ell \in \pi_{w+1}(Q_{n,r})$.

But since $D = \langle X \rangle = \langle f_1 \rangle \times \langle X_1 \rangle \times \ldots \times \langle X_r \rangle$ by Theorem 4.23

and Lemma 4.21 , it follows that if $\ell \notin \{i_1, \ldots, i_s\}$ then $h_\ell = 1$, and if $\ell = i_{s'} \in \{i_1, \ldots, i_s\}$ then

$$h_\ell = h_{i_{s'}} = x_{i_{s'},j_{s'}}^{t_{i_{s'},j_{s'}}} \in \pi_w(Q_{n,r}) \setminus \pi_{w+1}(Q_{n,r})$$

by hypothesis, which is a contradiction. Hence h also has cpp-weight w .

Let $i = p^m$ for some m such that $0 \leq m \leq n-1$. Suppose for contradiction that $f_1^{p^m} h$ has cpp-weight strictly greater than p^m. Then by the same argument as above, $f_1^{p^m} h = h_o h'$ where

$$h_o = f_1^{a_0 + a_1 p + \ldots + a_{n-1} p^{n-1}} ,$$

and if q is the smallest integer such that $a_q \neq 0$ then $p^q > p^m$ implies $q > m$; ....(9)

and $h' \in \langle X_j \; : \; j = 1, \ldots, r\rangle \cap \pi_{p^m+1}(Q_{n,r})$. Again, since $D = \langle f_1 \rangle \times \langle X_1, \ldots, X_r \rangle$ it follows that $f_1^{p^m} = h_o$ which contradicts (9) , and so $f_1^{p^m} h$ has cpp-weight p^m . □

<u>4.34 COROLLARY</u>

Let h in D be written in the form given by Corollary 4.25 . Then the cpp-weight of h is the minimum of the cpp-weights of those $x_{i,j}^{t_{i,j}}$ for which $t_{i,j} \neq 0$ in the expression for h given by Corollary 4.25 .

<u>Proof</u>

Suppose the minimum of the cpp-weights of those $x_{i,j}^{t_{i,j}}$ for which $t_{i,j} \neq 0$ in h is w . Let h' be the product of those $x_{i,j}^{t_{i,j}}$'s in h of cpp-weight w . Note that no more than one $x_{i,j}$ in X_i

contributes to h', by Lemma 4.32, and so by Lemma 4.33, h' has cpp-weight w. But by choice of h', $h = h'\underline{h}$ where $\underline{h}$ is in $\pi_{w+1}(Q_{n,r})$, and so h is in $\pi_w(Q_{n,r}) \setminus \pi_{w+1}(Q_{n,r})$, i.e. h has cpp-weight w as required. $\square$

<u>Proof of Lemma 4.31</u>

Let the set of those $x_{k,\ell}$ in X of cpp-weight i be $X(i)$, and suppose for contradiction that $\langle X(i), \pi_{i+1}(Q_{n,r}) \cap D \rangle$ is a proper subgroup of $\pi_i(Q_{n,r}) \cap D$. Then there exist a finite number of elements in D, say $g_1, \dots, g_m$, such that

$$\pi_i(Q_{n,r}) \cap D = \langle g_1, \dots, g_m, X(i), \pi_{i+1}(Q_{n,r}) \cap D \rangle,$$

and for $j = 1, \dots, m$, $g_j \notin \langle X(i), \pi_{i+1}(Q_{n,r}) \cap D \rangle$. Then if we express say g_1 in the form given by Corollary 4.25, g_1 must contain a product $\underline{h}$ of $x_{i,j}$'s each of cpp-weight strictly less than i. Let the minimum of the cpp-weights of the $x_{i,j}$'s in $\underline{h}$ be w, so $w < i$, and let h' be the product of those $x_{i,j}^{t_{i,j}}$'s in $\underline{h}$ of cpp-weight w, where $x_{i,j}^{t_{i,j}}$ is the power of $x_{i,j}$ in $\underline{h}$. Then by Corollary 4.34, h' is of cpp-weight w and so $\underline{h} \in \pi_w(Q_{n,r}) \setminus \pi_{w+1}(Q_{n,r})$. But then by definition of $\underline{h}$, $g_1 \in \pi_w(Q_{n,r}) \setminus \pi_{w+1}(Q_{n,r})$, so $g_1 \notin \pi_i(Q_{n,r})$, which is a contradiction, and we have the result. $\square$

To calculate the cpp-weights of elements in X, we need to look at powers of elements in X. For by Lemma 1.2, if $x_{k,\ell} \in X$ has cpp-weight w, then $x_{k,\ell}$ is a product, $\underline{x}_{k,\ell}$ say, of elements of the form g^{p^v} where $g \in \gamma_u(Q_{n,r}) \cap D$ and $up^v \geq w$. As we saw in the proof of Lemma 4.33, this implies $\underline{x}_{k,\ell}$ is a product of powers of elements in X_k if $\ell \neq 1$, and if $x_{k,\ell} = x_{r,1} = f_1$ then $\underline{x}_{k,\ell}$ is a product of powers of elements in $\langle f_1 \rangle$. We will first calculate the nilpotency weights of the powers of elements in X, which will give us a way of determining which powers of which elements in X_k are in the product $\underline{x}_{k,\ell}$, by comparison of nilpotency weights. We require a result about the orders of elements in X:

4.35 LEMMA

Let $i \in \{1, \ldots, r\}$, and let $j = p^{r-i}\{1 + k(p-1)\} + \ell$, where $k \in \{0, \ldots, n-1\}$ and $\ell \in \{1, \ldots, p^{r-i}(p-1)\}$. Then $x_{i,j}$ has order p^{n-k}.

Proof

By construction of the $x_{i,j}$'s in X, for which see Definition 4.12, and by the same arguments as used for (6) of Chapter III,

$$x_{i,j} = [[f_1, {}_{k(p-1)+t(\ell)}y_i], {}_{w_{i+1}}y_{i+1}, \ldots, {}_{w_r}y_r]^{(-1)^s}$$

where $s \in \{0,1\}$, $t(\ell) \in \{0, \ldots, p-2\}$ and for $u = i+1, \ldots, r$, $w_u \in \{0, \ldots, p-1\}$. Hence

$$x_{i,j} = [f_1, {}_{k(p-1)+t(\ell)}\, y_i]^{(-1)^{s'}} g$$

where $s' \in \{0,1\}$ and g is a product of conjugates of $[f_1, {}_{k(p-1)+t(\ell)}y_i]$ with respect to $\langle y_{i+1}, \ldots, y_r \rangle$ which are distinct from $[f_1, {}_{k(p-1)+t(\ell)} y_i]$, i.e. $\sigma(g)$ is contained in $\{p^i+1, p^i+2, \ldots, p^r\}$. Hence since $D = (C_{p^n})^{(p^r)}$ is abelian, the order of $x_{i,j}$ is that of $[f_1, {}_{k(p-1)+t(\ell)} y_i]$. But $\langle f_1, y_i \rangle$ is isomorphic to C_{p^n} wr C_p , and so by Corollary 2.29 , $[f_1, {}_{k(p-1)+t(\ell)} y_i]$ has order p^{n-k} , as required. $\square$

4.36 LEMMA

Let $i \in \{1, \ldots, r\}$ and let $j \in \{p^{r-i}+1, \ldots, p^{r-i}\{1+(n-2)(p-1)\}\}$. Then $x_{i,j}^{\ p}$ has nilpotency weight $j+p^{r-i}(p-1)$, i.e.

$$x_{i,j}^{\ p} \in \gamma_{j+p^{r-i}(p-1)}(Q_{n,r}) \setminus \gamma_{j+p^{r-i}(p-1)+1}(Q_{n,r}) .$$

Proof

Let $j = p^{r-i}\{1+k(p-1)\}+\ell$, where $k \in \{0, \ldots, n-2\}$ and $\ell \in \{1, 2, \ldots, p^{r-i}(p-1)\}$. Note that by construction, $x_{i,j}$ has nilpotency weight j , i.e. $x_{i,j} \in \gamma_j(Q_{n,r}) \setminus \gamma_{j+1}(Q_{n,r})$.

By Lemma 4.35 , $x_{i,j}$ has order p^{n-k} , and so $x_{i,j}^{\ p}$ has order p^{n-k-1} . By Lemma 4.21 and Corollary 4.25 ,

$$x_{i,j}^{\ p} = \Pi \{ x_{i,u}^{\ t_{i,u}} : \ x_{i,u} \in X_i , \quad 0 \leqslant t_{i,u} < p \} ,$$

and suppose that the smallest integer such that $t_{i,u} \neq 0$ is u' . Then since D is abelian, and since by construction the elements of X_i have distinct nilpotency weights, with

$$x_{i,w'} = [x_{i,w}, b_1, \ldots, b_{w'-w}]$$

for $w' > w$, and some $b_1, \ldots, b_{w'-w} \in \langle y_1, \ldots, y_r \rangle$, $x_{i,u'}$

has both the same nilpotency weight and the same order as $x_{i,j}^{p}$.
Thus by Lemma 4.35 ,

$$p^{r-i}\{1+(k+1)(p-1)\}+1 \leq u' \leq p^{r-i}\{1+(k+2)(p-1)\} ,$$

and consequently,

$$(x_{i,p^{r-i}\{1+k(p-1)\}+1})^{p} \in \gamma_{p^{r-i}\{1+(k+1)(p-1)\}+1}(Q_{n,r}) ,$$

while

$$(x_{i,p^{r-i}\{1+(k+1)(p-1)\}})^{p} \notin \gamma_{p^{r-i}\{1+(k+2)(p-1)\}+1}(Q_{n,r}) .$$

Since

$$(x_{i,p^{r-i}\{1+(k+1)(p-1)\}})^{p}$$

$$= [x_{i,p^{r-i}\{1+k(p-1)\}+1}, b_1, \ldots, b_{p^{r-i}(p-1)-1}]^{p}$$ for some $b_1, \ldots, b_{p^{r-i}(p-1)-1}$ in $\langle y_i, \ldots, y_r \rangle$,

$$= [(x_{i,p^{r-i}\{1+k(p-1)\}+1})^{p}, b_1, \ldots, b_{p^{r-i}(p-1)-1}]$$ by Lemma 3.17 iii) , as D is abelian,

and since $p^{r-i}\{1+(k+2)(p-1)\} - p^{r-i}\{1+(k+1)(p-1)\} - 1$ is just $p^{r-i}(p-1) - 1$, the result now follows. □

4.37 COROLLARY

Let $i \in \{1, \ldots, r\}$, and let $j \in \{p^{r-i} + \ell : \ell = 1, \ldots, p^{r-i}(p-1)\}$. Let $m \in \{1, \ldots, n-1\}$. Then $x_{i,j}^{p^m}$ has nilpotency weight $j + m p^{r-i}(p-1)$.

Proof

We proceed by induction on m . The case $m=1$ is just Lemma 4.36,

$j \leqslant p^{r-i+1}$. Now suppose the result is true for $m-1$. Then by hypothesis $x_{i,j}^{p^{m-1}}$ has nilpotency weight $j+(m-1)p^{r-i}(p-1)$, i.e.

$$x_{i,j}^{p^{m-1}} \in \gamma_{j+(m-1)p^{r-i}(p-1)}(Q_{n,r}) \setminus \gamma_{j+(m-1)p^{r-i}(p-1)+1}(Q_{n,r}) .$$

By Lemma 4.21 and Corollary 4.25 ,

$$x_{i,j}^{p^{m-1}} = \Pi\{ x_{i,u}^{t_{i,u}} : x_{i,u} \in X_i , \ 0 \leqslant t_{i,u} < p \} .$$

Thus since the elements of X_i have distinct nilpotency weights by construction, the smallest integer u such that $t_{i,u} \neq 0$ in the above product is the nilpotency weight of $x_{i,j}^{p^{m-1}}$, i.e. $j+(m-1)p^{r-i}(p-1)$.

Now $x_{i,j}^{p^m} = (x_{i,j}^{p^{m-1}})^p = \Pi\{ (x_{i,u}^{t_{i,u}})^p : x_{i,u} \in X_i , 0 \leqslant t < p \}$.

By Lemma 4.36 , $(x_{i,j+(m-1)p^{r-i}(p-1)})^p$ belongs to $\gamma_{j+mp^{r-i}(p-1)}(Q_{n,r}) \setminus \gamma_{j+mp^{r-i}(p-1)+1}(Q_{n,r})$, and for $u >$ $j+(m-1)p^{r-i}(p-1)$,

$$x_{i,u}^{p} \in \gamma_{j+mp^{r-i}(p-1)+1}(Q_{n,r}) .$$

Hence $x_{i,j}^{p^m}$ has nilpotency weight $j+mp^{r-i}(p-1)$ as required for the induction. $\square$

4.38 LEMMA

Let $i \in \{1, \dots, r\}$ and let $j \in \{p^{r-i}+1, \dots, p^{r-i+1}\}$. Then $x_{i,j}^{p^{n-1}}$ has cpp-weight jp^{n-1} , i.e.

$$x_{i,j}^{p^{n-1}} \in \pi_{jp^{n-1}}(Q_{n,r}) \setminus \pi_{jp^{n-1}+1}(Q_{n,r}) .$$

Proof

Clearly $x_{i,j}^{p^{n-1}}$ has cpp-weight at least jp^{n-1} since $x_{i,j}$ has nilpotency weight j. By Corollary 4.37,

$$x_{i,j}^{p^{n-1}} \in \gamma_{j+(n-1)p^{r-i}(p-1)}(Q_{n,r}) \setminus \gamma_{j+(n-1)p^{r-i}(p-1)+1}(Q_{n,r}) .$$

Suppose $x_{i,j}^{p^{n-1}}$ has cpp-weight strictly greater than jp^{n-1}. We aim to show this implies $x_{i,j}^{p^{n-1}}$ is a product of elements in $\gamma_{j+(n-1)p^{r-i}(p-1)+1}(Q_{n,r})$, which is a contradiction.

By Lemma 1.2 and Lemma 4.21, $x_{i,j}^{p^{n-1}}$ is a product $\underline{x}_{i,j}$ of powers of elements in X_i such that each of these powers is of cpp-weight at least that of $x_{i,j}^{p^{n-1}}$, i.e. by assumption at least $jp^{n-1}+1$. Suppose the power of $x_{i,k} \in X_i$ in $\underline{x}_{i,j}$ is $x_{i,k}^{s_{i,k}}$, and suppose $s_{i,k} \neq 0$. Let v in $\{0, \ldots, n-1\}$ be such that p^v divides $s_{i,k}$ but p^{v+1} does not divide $s_{i,k}$. Then $kp^v \geq jp^{n-1}+1$.

Note now that in general $p^m \geq m(p-1)+1$: the result is clearly true for $m=1$, and if true for $m-1$ then $p^m = p(p^{m-1}) \geq p(m-1)(p-1)+p > m(p-1)+1$.

Let $j = p^{r-i} + \ell$, so $1 \leq \ell \leq p^{r-i}(p-1)$. Then

$$\begin{aligned} kp^v &\geq jp^{n-1}+1 = p^{n-1}(p^{r-i}+\ell)+1 \\ \Rightarrow k &\geq p^{n-1-v}(p^{r-i}+\ell)+p^{-v}, \\ &> p^{n-1-v}(p^{r-i}+\ell), \\ &\geq p^{n-1-v}.p^{r-i}+\ell, \end{aligned}$$

$$p^{n-1-v}.p^{r-i} + \ell \geq p^{r-i}\{1 + (n-1-v)(p-1)\} + \ell .$$

By Corollary 4.37, $x_{i,k}^{p^v} \in \gamma_{k+vp^{r-i}(p-1)}(Q_{n,r})$, and by the above,

$$k + v\,p^{r-i}(p-1) \geq p^{r-i}\{1 + (n-1)(p-1)\} + \ell + 1 ,$$

$$= j + (n-1)\,p^{r-i}(p-1) + 1 ,$$

which gives us the required contradiction, and thus the result. □

4.39 COROLLARY

Let $i \in \{1, \ldots, r\}$, and let $j \in \{p^{r-i}+1, \ldots, p^{r-i+1}\}$. Let $m \in \{0, \ldots, n-1\}$. Then $x_{i,j}^{p^m}$ has cpp-weight $j\,p^m$, i.e.

$$x_{i,j}^{p^m} \in \pi_{j\,p^m}(Q_{n,r}) \setminus \pi_{j\,p^m+1}(Q_{n,r}) .$$

Proof

Since $x_{i,j}$ has nilpotency weight j, we clearly have $x_{i,j}^{p^m}$ is in $\pi_{j\,p^m}(Q_{n,r})$. By Lemma 4.38,

$$(x_{i,j}^{p^m})^{p^{n-m-1}} = x_{i,j}^{p^{n-1}} \in \pi_{jp^{n-1}}(Q_{n,r}) \setminus \pi_{jp^{n-1}+1}(Q_{n,r}) ,$$

and so $x_{i,j}^{p^m} \notin \pi_{jp^m+1}(Q_{n,r})$, as required. □

4.40 LEMMA

Let $i \in \{1, \ldots, r\}$, and let $j = p^{r-i}\{1 + k(p-1)\} + \ell$, where $k \in \{0, \ldots, n-1\}$ and $\ell \in \{1, \ldots, p^{r-i}(p-1)\}$. Then $x_{i,j}$ has cpp-weight $p^k(p^{r-i} + \ell)$, i.e.

$$x_{i,j} \in \pi_{p^k(p^{r-i}+\ell)}(Q_{n,r}) \setminus \pi_{p^k(p^{r-i}+\ell)+1}(Q_{n,r}) .$$

Proof

For $m = 1, \dots, p^{r-i}\{p+(p-1)(n-1)\}$, by Lemma 4.21 and Theorem 4.23, $\{\gamma_m(Q_{n,r}) \cap \langle X_i\rangle\} / \{\gamma_{m+1}(Q_{n,r}) \cap \langle X_i\rangle\}$ is cyclic of order p, and by Theorem 4.23, is generated by $x_{i,m}\{\gamma_{m+1}(Q_{n,r}) \cap \langle X_i\rangle\}$.

Let $x_{i,j}$ have cpp-weight w. By Corollary 4.37,

$$(x_{i,p^{r-i}+\ell})^{p^k} \in \{\gamma_j(Q_{n,r}) \cap \langle X_i\rangle\} \setminus \{\gamma_{j+1}(Q_{n,r}) \cap \langle X_i\rangle\},$$

and so

$$x_{i,j} = (x_{i,p^{r-i}+\ell})^{tp^k} h$$

where $0 < t < p$, and $h \in \gamma_{j+1}(Q_{n,r}) \cap \langle X_i\rangle$. Now h is a product of $x_{i,u}$'s such that $u > j$, and by construction, for $u > j$, $x_{i,u}$ has cpp-weight greater than or equal to $w+1$, by Lemma 4.32. Hence h has cpp-weight at least $w+1$. Thus $(x_{i,p^{r-i}+\ell})^{p^k}$ has cpp-weight w. Hence by Corollary 4.39, $w = p^k(p^{r-i}+\ell)$ as required. □

4.41 THEOREM

The base group $D = (C_{p^n})^{(p^r)}$ of $Q_{n,r} = C_{p^n} \operatorname{wr}^{(p^r)} P_r$ is generated modulo $\pi_2(Q_{n,r}) \cap D$ by f_1, where f_1 is as defined in (4) of Chapter III.

For $j = 2, \dots, d(Q_{n,r})$, the group $\pi_j(Q_{n,r}) \cap D$ is generated modulo $\pi_{j+1}(Q_{n,r}) \cap D$ by those $x_{i,u}$ in X such that $j = p^k(p^{r-i}+\ell)$, $u = p^{r-i}\{1+k(p-1)\}+\ell$ for some k in $\{0, \dots, n-1\}$ and some ℓ in $\{1, \dots, p^{r-i}(p-1)\}$.

Furthermore, no proper subset of these $x_{i,u}$'s generates $\pi_j(Q_{n,r}) \cap D$ modulo $\pi_{j+1}(Q_{n,r}) \cap D$.

Proof

By Theorem 4.23 , D is generated modulo $\gamma_2(Q_{n,r}) \cap D$ by f_1 , and so the same holds for D modulo $\pi_2(Q_{n,r}) \cap D$. The result for $j = 2, \ldots, d(Q_{n,r})$ follows immediately from Lemma 4.31 and Lemma 4.40 . $\square$

We can give a simpler generating set for each of the terms $\pi_j(Q_{n,r}) \cap D$:

4.42 THEOREM

For $u = 1, \ldots, p^r$, let x_u be the simple commutator obtained by taking the first u entries in the commutator $\underline{g}_r$ of $Q_{n,r}$, where $\underline{g}_r$ is as defined in Definition 3.15 . Then for $j = 1, \ldots, d(Q_{n,r})$,

$$\pi_j(Q_{n,r}) \cap D = \left\langle \pi_{j+1}(Q_{n,r}) \cap D \, , \, x_u^{p^v} : \begin{array}{l} u \in \{1, \ldots, p^r\} , \\ v \in \{0, \ldots, n-1\} , \\ \text{and } j = up^v \end{array} \right\rangle ,$$

and no proper subset of these $x_u^{p^v}$'s generates $\pi_j(Q_{n,r}) \cap D$ modulo $\pi_{j+1}(Q_{n,r}) \cap D$.

Proof

Note $x_1 = f_1$, and for i in $\{1, \ldots, r\}$, if u is in $\{p^{r-i}+1, \ldots, p^{r-i+1}\}$, then $x_u = x_{i,u} \in \langle X_i \rangle$.

For $j = 1$ the result is clear by Theorem 4.41 . So suppose $j \geq 2$. If $\pi_j(Q_{n,r}) \cap \langle X_i \rangle \neq \langle 1 \rangle$, then by Theorem 4.41 , $\pi_j(Q_{n,r}) \cap \langle X_i \rangle$ is generated modulo $\pi_{j+1}(Q_{n,r}) \cap \langle X_i \rangle$ by those $x_{i,w}$ in X_i such that $j = p^k(p^{r-i} + \ell)$ and $w = p^{r-i}\{1 + k(p-1)\} + \ell$ for some k in $\{0, \dots, n-1\}$ and some ℓ in $\{p^{r-i}+1, \dots, p^{r-i+1}\}$. By Theorem 4.23 , $\{\gamma_w(Q_{n,r}) \cap \langle X_i \rangle\} / \{\gamma_{w+1}(Q_{n,r}) \cap \langle X_i \rangle\}$ is of order p , and so by Corollary 4.39 and 4.37,

$$x_{i,w} = (x_{i,p^{r-i}+\ell})^{tp^k} h$$

where $0 < t < p$ and $h \in \gamma_{w+1}(Q_{n,r}) \cap \langle X_i \rangle \leq \pi_{j+1}(Q_{n,r}) \cap \langle X_i \rangle$, by construction of X_i , since if $w' > w$, then $x_{i,w'}$ has greater cpp-weight than $x_{i,w}$. The result now follows. $\square$

We explain the diagram overleaf of the cpp-structure of D . By construction, for $i = 1, \dots, r$, if $x_{i,j} \in X_i$ and $x_{i,j}$ has cpp-weight w , i.e. $x_{i,j} \in \pi_w(Q_{n,r}) \setminus \pi_{w+1}(Q_{n,r})$, and if $j' > j$, and $x_{i,j'} \in X_i$, then $x_{i,j'} \in \pi_{w+1}(Q_{n,r})$.

$2p - 1 = p^0\{1 + (p-1)\} + p-1$, so $x_{r,2p-1}$ has cpp-weight
$p\{p^0 + p-1\} = p^2$, by Theorem 4.41 ,

$2p = p^0\{1 + 2(p-1)\} + 1$, so $x_{r,2p}$ has cpp-weight
$p^2\{p^0 + 1\} = 2p^2$, by Theorem 4.41 ;

$p^2 + p = p\{1 + (p-1)\} + p$, so x_{r-1,p^2+p} has cpp-weight
$p\{p + p\} = 2p^2$, by Theorem 4.41 ,

cpp-structure of D for $n \geq 3$, $r \geq 5$:

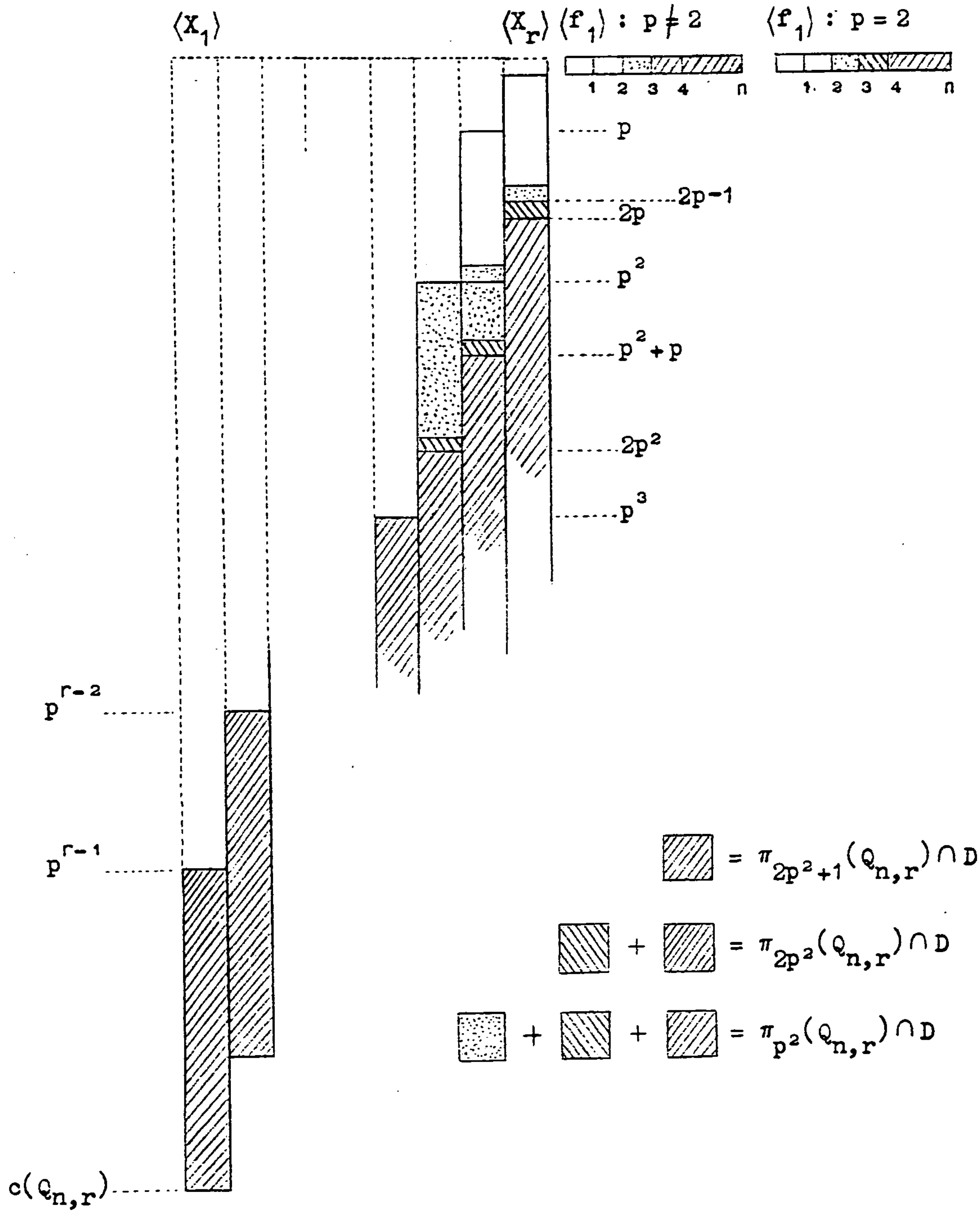


Fig. 9

and so on. If for i in $\{1, \ldots, r\}$, $x_{i,j}$ is in X_i, then $x_{i,j}$ has nilpotency weight j and so has cpp-weight at least j. Thus if $p^{r-i}+1 \geq k \geq 1$ for some k, $\langle X_i \rangle \leq \pi_k(Q_{n,r})$. In the diagram for $\langle f_1 \rangle$, the mth box contains $f_1^{p^{m-1}}$, for $m = 1, \ldots, n$; note that $f_1^{p^{m-1}}$ has cpp-weight p^{m-1}.

<u>4.43 COROLLARY</u>

For j in $\mathbb{Z}^+$,

$$\begin{aligned}\pi_j(Q_{n,r}) = \langle \pi_{j+1}(Q_{n,r}),\ z_{i,j},\ x_{i,j'} :\ & \text{for } i \in \{1, \ldots, r\},\ z_{i,j} \in \underline{Z},\\ & \text{and } x_{i,j'} \in \langle X_i \rangle, \text{ where}\\ & j' = p^{r-i}\{1 + k(p-1)\} + \ell \text{ and}\\ & j = p^k(p^{r-i} + \ell) \text{ for some } k \text{ in}\\ & \{0, \ldots, n-1\} \text{ and some } \ell \text{ in}\\ & \{1, \ldots, p^{r-i}(p-1)\}\rangle,\\ = \langle \pi_{j+1}(Q_{n,r}),\ z_{i,j},\ x_u^{p^h} :\ & \text{for } i \in \{1, \ldots, r\},\ z_{i,j} \in \underline{Z},\\ & u \in \{1, \ldots, p^r\} \text{ and } up^h = j\rangle,\end{aligned}$$

where $z_{i,j}$ and $\underline{Z}$ are as in Definition 4.7, x_u is as in Theorem 4.42, and X_i is as in Definition 4.15, and no proper subset of these generates $\pi_j(Q_{n,r})$.

<u>Proof</u>

The corollary follows directly from Corollary 4.30, Remark 4.10, and Theorems 4.41 and 4.42 . $\square$

It is now easy to calculate Shield's constant a for $Q_{n,r}$:

4.44 PROPOSITION

$$a(C_{p^n} \operatorname{wr}^{(p^r)} P_r) = 1 + (p-1)\left(\sum_{i=1}^{p^r} i\right)\left(\sum_{j=0}^{n-1} p^j\right) + (p-1)\sum_{k=0}^{r-1}\sum_{j=1}^{p^k} j ,$$

$$= 1 + \frac{(p^n-1)p^r(p^r+1)}{2} + \frac{p-1}{2}\left(\frac{p^r-1}{p-1} + \frac{p^{2r}-1}{p^2-1}\right) .$$

Proof

Recall by definition - see before Example 1.9 -

$$a(Q_{n,r}) = 1 + (p-1)\Sigma\{n(v) : 1 \leq v \leq d(Q_{n,r})\} ,$$

where $|\pi_v(Q_{n,r})| = p^{n(v)}$.

Let $D_0 = D$, and let $|\pi_v(Q_{n,r}) \cap D_i| = p^{n(i,v)}$ for $i = 0, \ldots, r$.

Then by Corollary 4.30 , Remark 4.10 and Lemma 4.2 , for $v = 1, \ldots, d(Q_{n,r})$,

$$|\pi_v(Q_{n,r})| = \prod_{i=0}^{r} |\pi_v(Q_{n,r}) \cap D_i| ,$$

$$= \prod_{i=0}^{r} p^{n(i,v)} .$$

Since D is abelian, by Corollary 4.43 each $x_u^{p^j}$ contributes 1 to each $n(0,v)$ for which $v \leq up^j$. Similarly, each $z_{i,j}$ contributes 1 to each $n(i,v)$ for which $v \leq j$. Hence $\Sigma\{n(\bar{v}) : 1 \leq v \leq d(Q_{n,r})\}$ is just the sum of the cpp-weights of $\{x_u^{p^j} : u = 1, \ldots, p^r , j = 0, \ldots, n-1\}$ and $\{z_{i,j} : i = 1, \ldots, r , j = 1, \ldots, p^{r-i}\}$, i.e.

$$a(Q_{n,r}) = 1 + (p-1)\left(\sum_{i=1}^{p^r} i\right)\left(\sum_{j=0}^{n-1} p^j\right) + (p-1)\sum_{k=0}^{r-1}\sum_{j=1}^{p^k} j \quad ,$$

as required. From this it is easy to obtain the second formulation by standard calculations. □

Since $P_r \cong Q_{1,r-1}$ we have

4.45 COROLLARY

$$a(P_r) = 1 + (p-1)\sum_{k=0}^{r-1}\sum_{j=1}^{p^k} j = 1 + \frac{p-1}{2}\left(\frac{p^r-1}{p-1} + \frac{p^{2r}-1}{p^2-1}\right) . \qquad \square$$

4.46 REMARK

The lower central and cpp-structure of $C_{p^n}\ \mathrm{wr}^{(p^r)} P_r$ is built up on the structure of C_{p^n} wr C_p . From Figure 4 , or Theorem 4.23 , recall that for i in $\{1, \ldots, r\}$ and j in $\{p^{r-i}+1, \ldots, c(Q_{n,r-i+1})\}$,

$$[\gamma_j(Q_{n,r}) \cap \langle X_i\rangle : \gamma_{j+1}(Q_{n,r}) \cap \langle X_i\rangle] = p .$$

We proved this for Theorem 4.23 by comparing the number $n(p^r-1)+1$ of elements in X , which indirectly uses the nilpotency class of $Q_{n,r}$ through the definition of X , with the order p^{np^r} of D , and by noting that $f_1^{p^{n-1}} \notin \gamma_2(Q_{n,r}) \cap D$, and this forces the result. A more fundamental reason is as follows. First note that for $2 \leq j \leq p+(p-1)(n-1) = c(C_{p^n}\ \mathrm{wr}\ C_p = Q_{n,1})$,

$$[\gamma_j(Q_{n,1}) \cap (C_{p^n})^{(p)} : \gamma_{j+1}(Q_{n,1}) \cap (C_{p^n})^{(p)}] = p \quad , \ldots\ldots(10)$$

since again for $\langle f_1\rangle = (C_{p^n})_1$, $f_1^{p^{n-1}} \notin \gamma_2(Q_{n,1})$ by

Corollary 4.19 , $|(C_{p^n})^{(p)}| = p^{np}$ and $c(Q_{n,1}) = p+(p-1)(n-1)$, which forces (10) . If $x_{i,j} \in X_i$ then by Lemma 4.20 ,

$$x_{i,j} = [f_1, {}_s y_i, b_1, \ldots, b_m]^t$$

where $0 < s \leq n(p-1)$, $0 < t < p^n$ and $b_1, \ldots, b_m$ are in $\{y_{i+1}, \ldots, y_r\}$. Then by Lemma 3.17 iii) , which we can apply since D is abelian,

$$x_{i,j}^{\,p} = [[f_1, {}_s y_i]^p, b_1, \ldots, b_m]^t \in \gamma_{j+1}(Q_{n,r})$$

by (10) , since $x_{i,j} \in \gamma_j(Q_{n,r})$ by definition.

A generalisation of $C_{p^n} \text{ wr}^{(p^r)} P_r = C_{p^n} \text{ wr } \underbrace{C_p \text{ wr } C_p \text{ wr } \ldots \text{ wr } C_p}_{r\ C_p\text{'s}}$

is $C_{p^{n_1}} \text{ wr } C_{p^{n_2}} \text{ wr } \ldots \text{ wr } C_{p^{n_{r+1}}}$, which we encountered in Theorem 3.4 . Let $\Lambda = (p^{n_2}) \times (p^{n_3}) \times \ldots \times (p^{n_{r+1}})$, the set on which $C_{p^{n_2}} \text{ wr } \ldots \text{ wr } C_{p^{n_{r+1}}}$ acts. Then it seems very likely that a generating set similar to X can be obtained for the base group $(C_{p^{n_1}})^\Lambda$ of $C_{p^{n_1}} \text{ wr } C_{p^{n_2}} \text{ wr } \ldots \text{ wr } C_{p^{n_{r+1}}}$, and that we would find that the lower central and cpp-structure of $C_{p^{n_1}} \text{ wr } \ldots \text{ wr } C_{p^{n_{r+1}}}$ within $(C_{p^{n_1}})^\Lambda$ is built up on that of $C_{p^n} \text{ wr } C_{p^r}$ for $r = n_2, \ldots, n_{r+1}$. It is not difficult to determine the generators of the lower central terms of $C_{p^n} \text{ wr } C_{p^r}$, which apart from b are just of the form $[f_1, {}_s b]$, where $\langle f_1 \rangle = (C_{p^n})_1$ and $\langle b \rangle = C_{p^r}$, by Corollary 3.18 and Lemma 1.1. So for $1 \leq j \leq p^{r-1}\{p+(p-1)(n-1)\} = c(C_{p^n} \text{ wr } C_{p^r})$, $\{\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}\} / \{\gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}\}$ is cyclic. Since $f_1^{p^{n-1}} \notin \gamma_2(C_{p^n} \text{ wr } C_{p^r})$ by Corollary 4.19, $(C_{p^n})^{(p^r)} / \{ \gamma_2(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} \}$ is of order p^n, but the other orders are more difficult to determine. We will give some of the constraints on these orders. □

<u>4.47 LEMMA</u>

Let $j \in \{ 1, \ldots, c(C_{p^n} \text{ wr } C_{p^r}) \}$. If

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}]$$

is p^s, then for $j < j' \in \{ 1, \ldots, c(C_{p^n} \text{ wr } C_{p^r}) \}$,

$$[\gamma_{j'}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_{j'+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}]$$

does not exceed p^s, i.e. the order of these factors decreases as j increases.

<u>Proof</u>

For $i = 1, \ldots, c(C_{p^n} \text{ wr } C_{p^r})$, $\gamma_i(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}$ is generated modulo $\gamma_{i+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}$ by $[f_1, {}_{i-1} b]$ where $\langle f_1 \rangle = (C_{p^n})_1$ and $\langle b \rangle = C_{p^r}$. Hence if $j' > j$,

$$\begin{aligned}
[f_1, {}_{j'-1} b]^{p^s} &= [[f_1, {}_{j-1} b]^{p^s}, {}_{j'-j} b] \quad \text{by Lemma 3.17 iii)} \\
&\qquad \text{since } (C_{p^n})^{(p^r)} \text{ is abelian,} \\
&\in \gamma_{j+1+j'-j}(C_{p^n} \text{ wr } C_{p^r}), \\
&= \gamma_{j'+1}(C_{p^n} \text{ wr } C_{p^r}),
\end{aligned}$$

and we obtain the result. □

4.48 LEMMA

$$[(C_{p^n})^{(p^r)} : \gamma_2(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}] = p^n ,$$

and for $j = 2, \ldots , c(C_{p^n} \text{ wr } C_{p^r})$,

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}]$$

does not exceed p^n .

Proof

By Corollary 3.18 , $(C_{p^n})^{(p^r)}$ is generated modulo $\gamma_2(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}$ by f_1 , and by Corollary 4.19 , $f_1^{p^{n-1}} \notin \gamma_2(C_{p^n} \text{ wr } C_{p^r})$. Hence

$$[(C_{p^n})^{(p^r)} : \gamma_2(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}] = p^n ,$$

and for $j \geqslant 2$ the result follows from Lemma 4.47 . $\square$

4.49 LEMMA

In C_{p^n} wr C_{p^r} let $\langle f_1 \rangle = (C_{p^n})_1$ and let $\langle b \rangle = C_{p^r}$. Then for $j = 2, \ldots , c(C_{p^n} \text{ wr } C_{p^r})$,

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}]$$

does not exceed p^r .

Proof

Since b has order p^r , the lemma follows immediately from the following result, Theorem 4.50 . $\square$

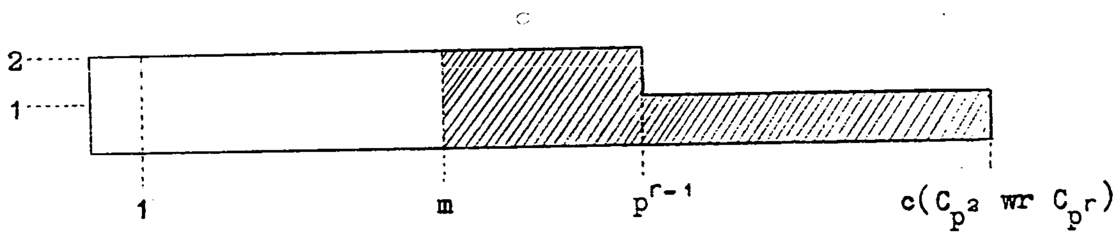
4.50 THEOREM (Theorem 2.85 [4])

Let G be a p-group, and let g be a complex commutator in G

of length w in its components $g_1, \dots, g_s$. Then the order of g modulo $\gamma_{w+1}(G)$ divides the order modulo $\gamma_2(G)$ of each g_i, $i = 1, \dots, s$. $\square$

We now have enough to deduce completely the orders of the factors $\gamma_j(C_{p^n} \text{ wr } C_{p^r}) / \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r})$ for $j = 1, \dots, c(C_{p^n} \text{ wr } C_{p^r})$ for certain values of n and r, i.e. $n = 2$ or $r = 2$.

<u>$(C_{p^2})^{(p^r)} \leqslant C_{p^2} \text{ wr } C_{p^r}$</u> :



$= \gamma_{m+1}(C_{p^2} \text{ wr } C_{p^r}) \cap (C_{p^2})^{(p^r)}$

Fig. 10

<u>4.51 COROLLARY</u>

For $j = 1, \dots, p^{r-1}$,

$$[\gamma_j(C_{p^2} \text{ wr } C_{p^r}) \cap (C_{p^2})^{(p^r)} : \gamma_{j+1}(C_{p^2} \text{ wr } C_{p^r}) \cap (C_{p^2})^{(p^r)}]$$

is p^2, and for $j = p^{r-1}+1, \dots, c(C_{p^2} \text{ wr } C_{p^r})$,

$$[\gamma_j(C_{p^2} \text{ wr } C_{p^r}) \cap (C_{p^2})^{(p^r)} : \gamma_{j+1}(C_{p^2} \text{ wr } C_{p^r}) \cap (C_{p^2})^{(p^r)}]$$

is p .

Proof

The order of $(C_{p^2})^{(p^r)}$ is p^{2p^r}, while the nilpotency class of C_{p^2} wr C_{p^r} is $p^{r-1}\{p+(p-1)(2-1)\} = 2p^r - p^{r-1}$ by Corollary 2.30 . The result now follows from Lemmas 4.47 and 4.48 . $\square$

$(C_{p^n})^{(p^2)} \leq C_{p^n}$ wr C_{p^2} :

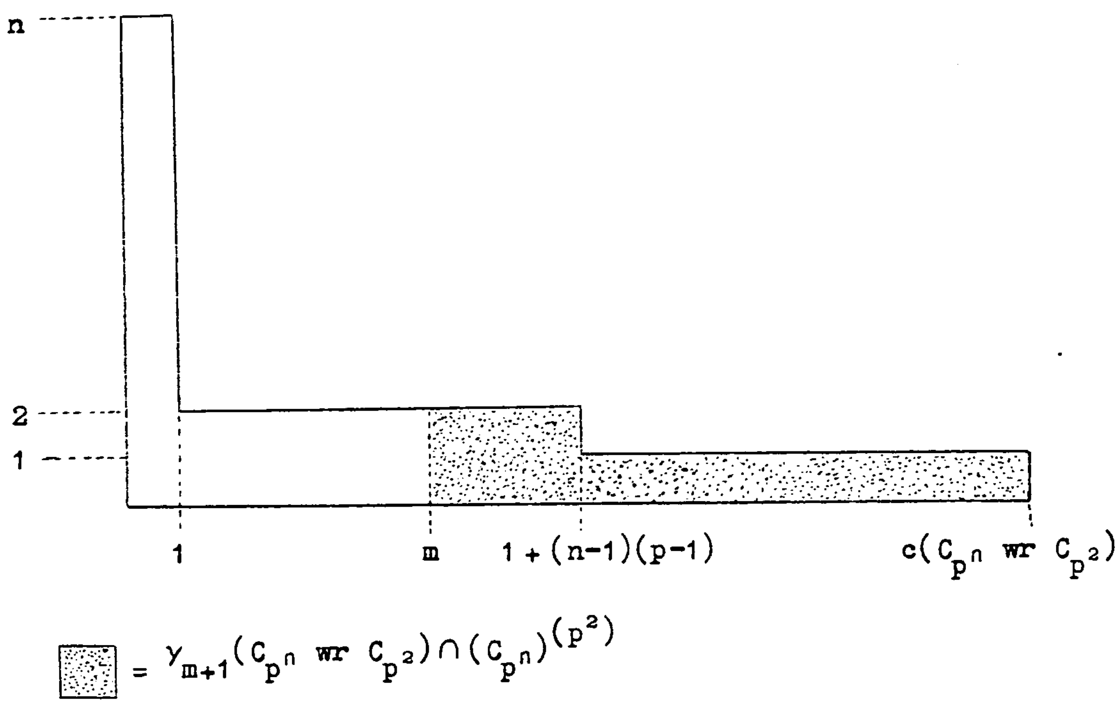


Fig. 11

4.52 COROLLARY

$$[(C_{p^n})^{(p^2)} : \gamma_2(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)}] = p^n ;$$

for $j = 2, \ldots, 1+(p-1)(n-1)$,

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)}]$$

is p^2; and for $j = (p-1)(n-1)+2, \ldots, c(C_{p^n} \text{ wr } C_{p^2})$,

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)}]$$

is p.

<u>Proof</u>

The first part follows from Lemma 4.48. Thus since $(C_{p^n})^{(p^2)}$ has order p^{np^2}, we have from this first part that

$$|\gamma_2(C_{p^n} \text{ wr } C_{p^2}) = \gamma_2(C_{p^n} \text{ wr } C_{p^2}) \cap (C_{p^n})^{(p^2)}| = p^{n(p^2-1)}.$$

By Lemma 4.49, none of the factors for $j \geq 2$ has order exceeding p^2, and so the number of factors of order p^2 is

$$\begin{aligned} & n(p^2-1) - (c(C_{p^n} \text{ wr } C_{p^2}) - 1) \\ &= n(p^2-1) + 1 - p\{p + (p-1)(n-1)\}, \\ &= np^2 - (n-1) - p - np(p-1), \\ &= (n-1)(p-1), \end{aligned}$$

and the result now follows from Lemma 4.47. □

<u>4.53 REMARK</u>

Recall that C_{p^n} wr C_{p^r} is a subgroup of C_{p^n} $\text{wr}^{(p^r)} P_r$. Hence for $j = 1, \ldots, c(Q_{n,r})$ $\big(= c(C_{p^n} \text{ wr } C_{p^r})$, by Theorem 3.3 and Corollary 2.30$\big)$,

$$|\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}| \leq |\gamma_j(Q_{n,r}) \cap (C_{p^n})^{(p^r)}|. \qquad \square$$

Remark 4.53 enables us to show that for $j \geq 2$, p^2 is not in general an upper bound for

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}],$$

as Corollaries 4.51 and 4.52 might suggest. In fact, the upper bound p^r for $j \geq 2$ is a best general bound :

4.54 LEMMA

For n sufficiently large ,

$$[\gamma_2(C_{p^n} \text{ wr } C_{p^r}) : \gamma_3(C_{p^n} \text{ wr } C_{p^r})]$$

$$= [\gamma_2(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)} : \gamma_3(C_{p^n} \text{ wr } C_{p^r}) \cap (C_{p^n})^{(p^r)}],$$

$$= p^r .$$

Proof

The first equality is general and trivial, following immediately from Lemma 2.6 i) : the results earlier in this section have been stated in the second form merely to emphasise we were looking at the base group.

Let n be such that $p + (p-1)(n-1) > p^r$. Then from Figure 7 ,

$$[\gamma_2(Q_{n,r}) \cap D : \gamma_{p+(p-1)(n-1)+1}(Q_{n,r}) \cap D] = p^m ,$$

where

$$m = r(p + (p-1)(n-1) - p^{r-1}) + (r-1)(p^{r-1} - p^{r-2}) + (r-2)(p^{r-2} - p^{r-3}) + \dots + (p-1) ,$$

$$= r(p + (p-1)(n-1)) - p^{r-1} - p^{r-2} - \dots - p - 1 ,$$

$$= r(p + (p-1)(n-1)) - \frac{p^r - 1}{p-1} .$$

By Remark 4.53 ,

$$[\gamma_2(C_{p^n} \text{ wr } C_{p^r}) : \gamma_{p+(p-1)(n-1)+1}(C_{p^n} \text{ wr } C_{p^r})] \geq p^m .$$

Suppose for contradiction that

$$[\gamma_2(C_{p^n} \text{ wr } C_{p^r}) : \gamma_3(C_{p^n} \text{ wr } C_{p^r})] \leq p^{r-1} , \qquad \ldots\ldots\ldots(11)$$

which implies for $j \geq 2$ that

$$[\gamma_j(C_{p^n} \text{ wr } C_{p^r}) : \gamma_{j+1}(C_{p^n} \text{ wr } C_{p^r})] \leq p^{r-1}$$

by Lemma 4.47 . Then

$$\begin{aligned}[\gamma_2(C_{p^n} \text{ wr } C_{p^r}) : \gamma_{p+(p-1)(n-1)+1}(C_{p^n} \text{ wr } C_{p^r})] &\leq (p^{r-1})^{n(p-1)}, \\ &= p^{n(r-1)(p-1)}.\end{aligned}$$

But we can choose n so large that $n(r-1)(p-1) < m$, which gives a contradiction to (11) :

$$\begin{aligned} & (r-1)\,n(p-1) < m \\ \Leftrightarrow\ & (r-1)\,n(p-1) < r(\,p + (p-1)(n-1)) - \frac{p^r - 1}{p-1} \\ & \qquad\qquad \text{by definition of } m , \\ \Leftrightarrow\ & (r-1)\,n(p-1) < r(\,1 + n(p-1)) - \frac{p^r - 1}{p-1} , \\ \Leftrightarrow\ & n(p-1) > \frac{p^r - 1}{p-1} - r , \\ \Leftrightarrow\ & n > \frac{p^r - 1}{(p-1)^2} - \frac{r}{p-1} . \end{aligned}$$

Hence we obtain the result. $\square$

CHAPTER V : A second proof of the nilpotency class of $C_{p^n} wr^{(p^r)} P_r$.

Note that throughout this chapter we take $Q_{n,r} = C_{p^n} wr^{(p^r)} P_r =$

$C_{p^n}\ wr\ \underbrace{C_p\ wr\ C_p\ wr \ldots C_p}_{r\ C_p\text{'s}}$.

Note also we aim not to use Shield's results as aids to proofs.

We begin with a lemma which allows us to "change" the order of entries in a simple commutator.

5.1 LEMMA

Let A be an abelian group, and let "Λ , B" be a pair. Let $W = A\ wr^{\Lambda} B$, let $g \in A^{\Lambda}$ and $x, z \in B$. Then

i) $[g, z, x] = [g, x, z][g^{xz}, [z, x]]$.

Furthermore, if $[z, x]$ has finite order t then $[g^{xz}, [z, x]]$ can be written as a product of commutators of the form

ii) $[g^{xz}, {}_s[x, z]]$, where $1 \leq s \leq t-1$.

Proof

i) $[g, z, x] = [g^{-1}g^{z}, x]$,

$= [g^{-1}, x][g^{z}, x]$ by Lemma 3.17 ii) ,

$= g\, g^{-x}\, g^{-z}\, g^{zx}$,

$= \underbrace{g\, g^{-x}\, g^{-z}\, g^{xz}}\, g^{-xz}\, g^{zx}$,

$= [g, x, z]\, g^{-xz}\, g^{xz[z, x]}$,

$= [g, x, z][g^{xz}, [z, x]]$.

ii) If $[z, x]$ has finite order t then since $[z, x] = [x, z]^{-1}$

$[g^{xz}, [z, x]] = [g^{xz}, [x, z]^{t-1}]$, to which we can apply Lemma 1.1 ii) repeatedly to obtain the required product. $\square$

Note that Lemma 5.1 i) is simply a statement for the group W in group-theoretic terms of a well-known integer group ring identity - see the proof of Lemma 3.4 [17] :

$$(z-1)(x-1) = (x-1)(z-1) + xz([z, x] - 1) ,$$

where x and z are elements of the group B whose integer group ring $\mathbb{Z}B$ is under consideration.

As a corollary we have a generalisation of Corollary 5.7 [9] to the permutational wreath product :

5.2 COROLLARY

If in addition B is abelian and $b_1', \ldots, b_m'$ is a permutation of the elements $b_1, \ldots, b_m$ of B then

$$[g, b_1, \ldots, b_m] = [g, b_1', \ldots, b_m']$$

for g in A^Λ, i.e. the two commutators represent the same element in $A \text{ wr}^\Lambda B$.

Proof

We proceed by induction on m . The case $m=2$ is just Lemma 5.1 , for which the second commutator $[g^{xz}, [z, x]] = 1$ since B is abelian.

Now suppose the result is true for sets of up to $m-1$ elements of B , and let $b_1', \ldots, b_m'$ be a permutation of the elements $b_1, \ldots, b_m$ of B . We treat seperately the cases $b_m' = b_1$ and $b_m' \neq b_1$.

i) $\underline{b'_m = b_1}$: By Lemma 5.1 , $[g, b_1, b_2] = [g, b_2, b_1]$. Let

$h = [g, b_2]$, which is in A^Λ . Then

$$[g, b_1, \ldots, b_m] = [h, \underbrace{b_1, b_3, \ldots, b_m}_{m-1}] ,$$

$$= [h, b_m, b_3, \ldots, b_{m-1}, b_1]$$

by the induction hypothesis ,

$$= [[g, \underbrace{b_2, b_m, b_3, \ldots, b_{m-1}}_{m-1}], b_1] ,$$

$$= [[g, b'_1, \ldots, b'_{m-1}], b_1]$$

by the induction hypothesis ,

$$= [g, b'_1, \ldots, b'_m] \quad \text{as required.}$$

ii) $\underline{b'_m \neq b_1}$: We have that for some i in $\{2, \ldots, m\}$, $b'_m = b_i$.

Let $h = [g, b_1]$, which is in A^Λ . Then by the induction hypothesis,

$$[h, b_2, \ldots, b_m] = [h, b_2, \ldots, b_{i-1}, b_m, b_{i+1}, \ldots, b_{m-1}, b_i] ,$$

$$= [h, b_2, \ldots, b_{i-1}, b_m, b_{i+1}, \ldots, b_{m-1}, b'_m] ,$$

i.e. $b_i\,(=b'_m)$ and b_m have exchanged places.

Now $[h, b_2, \ldots, b_{i-1}, b_m, b_{i+1}, \ldots, b_{m-1}]$

$$= [g, \underbrace{b_1, b_2, \ldots, b_{i-1}, b_m, b_{i+1}, \ldots, b_{m-1}}_{m-1}] ,$$

$$= [g, b'_1, \ldots, b'_{m-1}] \quad \text{by the induction hypothesis,}$$

and so $[g, b_1, \ldots, b_m] = [g, b'_1, \ldots, b'_m]$ as required. $\square$

5.3 COROLLARY

Let A be an abelian group, and let "A , B" be a pair. Let

$W = A \operatorname{wr}^{\Lambda} B$, let $g \in A^{\Lambda}$ and $b_1, \dots, b_m \in B$. Then for m in $\mathbb{N}$, $m \geq 3$,

$$[g, b_1, \dots, b_m]$$

$$= [g, b_2, \dots, b_m, b_1][g^{b_2 b_1}, [b_1,b_2], b_3, \dots, b_m] \times$$

$$\prod_{i=2}^{m-1} \Big[[g, b_2, \dots, b_i]^{b_{i+1} b_1}, [b_1,b_{i+1}], b_{i+2}, \dots, b_m\Big] .$$

<u>Proof</u>

We proceed by induction on m .

<u>m=3</u> : $[g, b_1, b_2, b_3] = [g, b_2, b_1, b_3][g^{b_2 b_1}, [b_1,b_2], b_3]$ by Lemma 5.1 . Let $h = [g, b_2]$, ~~which is~~ in A^{Λ}. Then

$$[g, b_2, b_1, b_3] = [h, b_1, b_3] ,$$

$$= [h, b_3, b_1][h^{b_3 b_1}, [b_1,b_3]] ,$$

by Lemma 5.1 . Hence since all commutators starting with g in A^{Λ} are in A^{Λ} , which is abelian, we obtain the result for $m=3$.

Now suppose the result is true for $3, \dots, m$. Then by hypothesis,

$$[g, b_1, \dots, b_{m+1}] = [h_1 h_2 h_3, b_{m+1}]$$

where

$$h_1 = [g, b_2, \dots, b_m, b_1] ,$$

$$h_2 = [g^{b_2 b_1}, [b_1,b_2], b_3, \dots, b_m] , \text{ and}$$

$$h_3 = \prod_{i=2}^{m-1} \Big[[g, b_2, \dots, b_i]^{b_{i+1} b_1}, [b_1,b_{i+1}], b_{i+2}, \dots, b_m\Big] .$$

Since all commutators starting with g in A^{Λ} are in A^{Λ} , which is abelian, we can apply Lemma 3.17 iii) to obtain

$$[h_1 h_2 h_3, b_{m+1}] = [h_1, b_{m+1}][h_2, b_{m+1}][h_3, b_{m+1}] .$$

Now $[h_2, b_{m+1}]$ is the second required commutator for $m+1$. Since $[g, b_2, \ldots, b_m] \in A^{\Lambda}$, we have by Lemma 5.1 ,

$$[h_1, b_{m+1}]$$

$$= [g, b_2, \ldots, b_{m+1}, b_1]\left[[g, b_2, \ldots, b_m]^{b_{m+1}b_1}, [b_1, b_{m+1}]\right] ,$$

the first required commutator and the factor for $i=m$ in the required product for $m+1$. Since $h_3 \in A^{\Lambda}$, by Lemma 3.17 iii) ,

$$[h_3, b_{m+1}]$$

$$= \prod_{i=2}^{m-1} \left[[g, b_2, \ldots, b_i]^{b_{i+1}b_1}, [b_1, b_{i+1}], b_{i+2}, \ldots, b_{m+1}\right] ,$$

all but the factor for $i=m$ in the required product for $m+1$. Thus, since $[h_1h_2h_3, b_{m+1}] \in A^{\Lambda}$, which is abelian, we can rearrange the commutators obtained above, and the result for $m+1$ follows.

Induction now yields the corollary. □

We will see that Lemma 5.1 and its corollaries enable us to re-express any simple commutator in $Q_{n,r}$ with first entry from $D = (C_{p^n})^{(p^r)}$ and other enties from P_r as a product of simple commutators with first entry f_1 , where $\langle f_1 \rangle = (C_{p^n})_1$, and other entries from $\{z_1, \ldots, z_r\}$ where the z_i's are as defined in Definition 4.6 . A lemma towards this result is

5.4 LEMMA

Let $\underline{g} \neq 1$ be a simple commutator of length k in $Q_{n,r}$, with first entry g from $(C_{p^n})^{(p^r)}$ and other entries from the set

$\{y_1, \ldots, y_r\}$. Then g can be re-expressed as a product of commutators each of length at least k and of the form

$$[g^z, b_{1,1}, b_{1,2}, \ldots, b_{1,\ell_1}, b_{2,1}, \ldots, b_{2,\ell_2}, \ldots, b_{r-1,\ell_{r-1}}, {}_{\ell_r}y_r], \qquad \ldots\ldots\ldots(1)$$

where g^z is a conjugate of g , $\ell_r \geq 0$, and such that for $i = 1, \ldots, r-1$,

i) if $\ell_i = 0$ then there are no terms in $b_{i,j}$ in (1) ;

ii) for $1 \leq j \leq \ell_i$, $b_{i,j}$ is a simple commutator of length at least 1 which starts with y_i and has other entries from $\{y_{i+1}, \ldots, y_r\}$;

and iii) for fixed i , permuting the $b_{i,j}$'s in (1) produces a commutator which represents the same element as (1) .

<u>Proof</u>

Note that ii) implies $b_{i,j} \in D_i$, where D_i is as defined in (1) of Chapter IV .

<u>Part iii)</u> : This follows easily from Corollary 5.2 , since D_i is abelian and $[g^z, b_{1,1}, \ldots, b_{i-1,\ell_{i-1}}]$ is in $(C_{p^n})^{(p^r)}$, the abelian base group of $Q_{n,r}$.

Before giving the proof of the main part of the lemma, which consists of a double inverse induction, we work through an example to illustrate the method of the proof, to try to make the proof easier to follow. Note that we will use Lemma 4.8 about the structure of P_r . Recall Lemma 4.8 was deduced from work of Weir, for which Shield's results were not used.

5.5 EXAMPLE

We wish to express $\underline{g} = [\, g, y_2, y_3, y_3, y_1 \,] \in C_{p^n} wr^{(p^3)} P_3$ as a product of commutators each of length at least 5 of the form (1) .

a) We first want to "shift" the second, i.e. the last, y_3 to the extreme right of $\underline{g}$. Note we underline y_3's which come from this second y_3 to mark what happens to it during shifting.

Step 1 : By Lemma 5.1 , since $[\, g, y_2, y_3 \,] \in (C_{p^n})^{(p^3)}$,

$$\begin{aligned} \underline{g} &= [\, g, y_2, y_3, \underline{y}_3, y_1] \,, \\ &= [\, g, y_2, y_3, y_1, \underline{y}_3] \Big[[\, g, y_2, y_3 \,]^{y_1 \underline{y}_3}, [\, \underline{y}_3, y_1 \,] \Big] \,, \\ &= h_1 h_2 \,, \quad \text{say} . \end{aligned}$$

Step 2 : $[\underline{y}_3, y_1 \,] = [\, y_1, \underline{y}_3 \,]^{-1} = [\, y_1, \underline{y}_3 \,]^{p-1}$, since $[\, y_1, y_3 \,]$ is a product of conjugates of y_1 which commute and are each of order p . Thus

$$h_2 = \Big[[\, g, y_2, y_3 \,]^{y_1 \underline{y}_3}, [\, y_1, \underline{y}_3 \,]^{p-1} \Big] \,.$$

Step 3 : $[\, g, y_2, y_3 \,]^{y_1 \underline{y}_3} = [\, g^{y_1 \underline{y}_3}, y_2^{\,y_1 \underline{y}_3}, y_3^{\,y_1 \underline{y}_3} \,]$.

$$\begin{aligned} \text{Now} \quad y_2^{\,y_1 \underline{y}_3} &= y_2 [\, y_2, y_1 \underline{y}_3 \,] \,, \\ &= y_2 [\, y_2, \underline{y}_3 \,] [\, y_2, y_1 \,] [\, y_2, y_1, \underline{y}_3 \,] \quad \text{by Lemma 1.1 ii)}, \\ &= y_2 [\, y_2, \underline{y}_3 \,] [\, y_1, y_2 \,]^{-1} \Big[[\, y_1, y_2 \,]^{-1}, \underline{y}_3 \Big] \quad \text{by Step 2}, \\ &= y_2 [\, y_2, \underline{y}_3 \,] [\, y_1, y_2 \,]^{p-1} \Big[[\, y_1, y_2 \,]^{p-1}, \underline{y}_3 \Big] \\ &\qquad \text{by Step 2}, \\ &= y_2 [\, y_2, \underline{y}_3 \,] [\, y_1, y_2 \,]^{p-1} [\, y_1, y_2, \underline{y}_3 \,]^{p-1} \end{aligned}$$

since $[y_1, y_2]^{p-1} = (y_1^{-1} y_1^{y_2})^{p-1} \in D_1$ which is abelian, and now apply Lemma 3.17 ii) . This is an explicit, computational method for obtaining $[y_2, y_1, y_3]$ as a product of simple commutators each of length at least 3 which start with y_1 and have other entries from $\{y_2, y_3\}$. Alternatively, note that $[y_2, y_1, y_3]$ is in D_1, since it contains $y_1 \in D_1$ and $1 < 2,3$, and so by Lemma 4.8 we know there exists a new expression for $[y_2, y_1, y_3]$ which is a product of simple commutators each of length at least 3 starting with y_1 and with other entries from $\{y_2, y_3\}$. Either way, we have shown that $y_2^{y_1 \underline{y}_3}$ is a product of simple commutators each of which starts with a y_i for some i in $\{1, 2\} \subseteq \{1, 2, 3\}$ and has other entries from $\{y_{i+1}, \dots, y_3\}$, but none start with the y_3 being shifted, $\underline{y}_3$.

$$\begin{aligned} \text{Similarly,}\quad y_3^{y_1 \underline{y}_3} &= y_3 [y_3, y_1 \underline{y}_3] \;, \\ &= y_3 [y_3, \underline{y}_3][y_3, y_1][y_3, y_1, \underline{y}_3] \;, \end{aligned}$$

and as above we can express $y_3^{y_1 \underline{y}_3}$ as a product of simple commutators with first entry y_i for some i in $\{1, 2, 3\}$ and other entries from $\{y_{i+1}, \dots, y_3\}$, but none starting with $\underline{y}_3$.

<u>Step 4</u> : Re-express $y_2^{y_1 \underline{y}_3}$ and $y_3^{y_1 \underline{y}_3}$ in

$$h_2 = \left[\, g^{y_1 \underline{y}_3}, y_2^{y_1 \underline{y}_3}, y_3^{y_1 \underline{y}_3}, [y_1, \underline{y}_3]^{p-1} \right] ,$$

as given in Step 3 , and apply Lemma 1.1 ii) repeatedly to obtain a new expression for h_2 as a product of commutators of the form

$$[\, g^{y_1 \underline{y}_3}, b_1, \dots, b_v] \qquad \dots\dots.(2)$$

where each b_j is a simple commutator, of length k_j, say, with first entry y_i for some i in $\{1, 2, 3\}$ and other entries from $\{y_{i+1}, \ldots, y_3\}$. Note however that no b_j starts with $\underline{y}_3$, which we can think of as having been "absorbed" by the b_j's. Then the length of (2) regarded as a complex commutator is

$$\sum_{j=1}^{v} k_j + 1 \text{ , which is greater than } 5 \text{ .}$$

Since $\underline{y}_3$ is in the last position of h_1 and has been absorbed by h_2, we have finished shifting this $\underline{y}_3$.

b) We must now shift the second last y_3 in $\underline{g}$ by applying the same method as in a) to h_1, and to the product of commutators of the form (2) making up h_2. We then end up with a new expression for $\underline{g}$, as $\underline{g}'$, a product of commutators each of length at least 5 and of the form

$$[\, g^z, b_1, \ldots, b_v, {}_{\ell_3}y_3 \,] \qquad \ldots\ldots.(3)$$

where g^z is a conjugate of g, and is thus in $(C_{p^n})^{(p^3)}$, $0 \leq \ell_3 \leq 2$, each b_j is a simple commutator with first entry y_i for some i in $\{1, 2\}$ and other entries from $\{y_{i+1}, \ldots, y_3\}$, and v is not necessarily as in (2).

c) We now want to shift all b_i's, in each commutator of type (3) in $\underline{g}'$, which start with y_2. We proceed in the same way as for the y_3's, starting with the last such b_i, but note we have to use Lemma 4.8 rather than the first method described in Step 3, as b_i may not be equal to y_2 : for example,

$$\left[g, [y_2,y_3], [y_1,y_2]\right]$$

$$= \left[g, [y_1,y_2], [y_2,y_3]\right]\left[g^{[y_1,y_2][y_2,y_3]}, \left[[y_2,y_3], [y_1,y_2]\right]\right],$$

and $\left[[y_2,y_3], [y_1,y_2]\right] = \left[[y_1,y_2], [y_2,y_3]\right]^{-1}$ is not simple, but does belong to D_1 since it contains y_1 and $1 < 2,3$. We want to re-express this as a product of simple commutators each of length at least 4 with first entry y_1 and other entries from $\{y_2, y_3\}$: Lemma 4.8 tells us it is possible to do this, even though no explicit re-expression is given.

Thus we finally end up with a new expression for $\underline{g}$ as a product of commutators each of length at least 5 and of the form

$$[g^z, b_{1,1}, b_{1,2}, \ldots, b_{1,\ell_1}, b_{2,1}, \ldots, b_{2,\ell_2}, \ell_3 y_3] \qquad \ldots\ldots.(4)$$

where g^z is a conjugate of g ; $b_{1,1}, \ldots, b_{1,\ell_1}$ are simple commutators which start with y_1 , and have other entries from $\{y_2, y_3\}$; $b_{2,1}, \ldots, b_{2,\ell_2}$ are simple commutators which start with y_2 and have other entries equal to y_3 ; and $0 \leqslant \ell_3 \leqslant 2$, i.e. we have expressed $\underline{g}$ in the required form. $\square$

<u>Proof of Lemma 5.4 con.</u>

Note that y_r is of the form $b_{r,j}$, i.e. a simple commutator which starts with y_r and has other entries from $\{y_{r+1}, \ldots, y_r\}$, which is the empty set.

The outer inverse induction is on the indices of the y_i's , starting with r : we aim to show that when we have "shifted" all elements $y_i, \ldots, y_r$ in $\underline{g}$ to the right, we will have re-expressed $\underline{g}$ as $\underline{g}'$, a product of commutators each of length at least k and of the form

$$[g^z, b_1, \ldots, b_u, b_{i,1}, \ldots, b_{i,\ell_i}, \ldots, b_{r-1,\ell_{r-1}}, {}_{\ell_r} y_r] \qquad \ldots\ldots.(5)$$

where g^z is a conjugate of g ; $b_1, \ldots, b_u$ are simple commutators which start with y_j for some j in $\{1, \ldots, i-1\}$ and other entries from $\{y_{j+1}, \ldots, y_r\}$; each $b_{i,j}$ is as described in the statement of the lemma, and $\ell_r \geqslant 0$. Note that (5) corresponds to (3) in Example 5.5 .

The inner inverse induction is on the maximum number of b_j's starting with y_{i-1} in each of the commutators of type (5) making up $\underline{g}'$, for $i = 2, \ldots, r+1$: let k_{i-1} be this maximum . In other words, each of the commutators of type (5) in $\underline{g}'$ has at most k_{i-1} b_j's starting with y_{i-1} , for $i = 2, \ldots, r+1$. Then when we have shifted the last t of these b_j's to the right in each of the commutators (5) we will have re-expressed $\underline{g}$ as a product of commutators each of length at least k and of the form

$$[g^{z'}, b'_1, \ldots, b'_{u'}, b_{i-1,s}, \ldots, b_{i-1,\ell_{i-1}}, b_{i,1}, \ldots, b_{r-1,\ell_{r-1}}, {}_{\ell} y_r] \qquad \ldots\ldots.(6)$$

where $g^{z'}$ is a conjugate of g ; $b'_1, \ldots, b'_{u'}$ are simple commutators which start with y_j for some j in $\{1, \ldots, i-1\}$ and have other entries from $\{y_{j+1}, \ldots, y_r\}$, and at most $k_{i-1} - t$ of these start with y_{i-1} ; $0 \leqslant \ell_{i-1} - s + 1 \leqslant t$; $b_{i,1}, \ldots, b_{r-1,\ell_{r-1}}$ are as in (5) ; and $\ell = \ell_r$ if $i-1 < r$.

Note that (6) corresponds to (2) and h_1 in Example 5.5 .

We may dispose of the cases where $\underline{g}$ has length at most 3 first. If $\underline{g}$ has length 1 or 2 there is nothing to prove. If $\underline{g}$ has length 3 then

$$\underline{g} = [\, g, y_i, y_j \,] \;, \quad \text{for some } i,j \text{ in } \{1, \dots, r\} \,,$$

and if $i \leq j$ there is again nothing to prove. If $i > j$, then since $[\, y_i, y_j \,] = y_j^{\,-y_i} y_j$ is of order p , we may apply Lemma 5.1 i), ii) to obtain the result.

From now on we assume $\underline{g}$ has length at least 4 . To start the double inverse induction we consider the case

$\underline{r, k_r}$: Note that k_r is the number of y_r's in $\underline{g}$. We may assume $k_r > 0$, for if the largest i such that $\underline{g}$ contains a y_i as an entry is less than r , we consider $\underline{g}$ as belonging to $C_p \cap wr^{(p^i)} P_i$, and work with $C_p \cap wr^{(p^i)} P_i$ instead, which is clearly just a change in notation from r to i in the following proof.

Our aim is to shift the last y_r in $\underline{g}$ to the right. If this y_r is already in the last position in $\underline{g}$ there is nothing to prove, since the stage r, k_r is trivially complete, so suppose the last y_r in $\underline{g}$ is not in the last position in $\underline{g}$. Let h be the value of the commutator $\underline{g}$ up to and including the entry before the last y_r , say

$$h = [\, g, b_1, \dots, b_q \,]$$

for some $b_1, \dots, b_q$ in $\{ y_1, \dots, y_r \}$, and let

$$\underline{g} = [\, h, y_r, b_{q+1}, \dots, b_{k-2} \,]$$

where $b_{q+1}, \dots, b_{k-2}$ are from the set $\{y_1, \dots, y_{r-1}\}$, by choice of y_r.

Step 1 : Note that $h \in (C_{p^n})^{(p^r)}$. If $q+1 = k-2$, by Lemma 5.1 i)

$$\underline{g} = [h, y_r, b_{k-2}],$$

$$= [h, b_{k-2}, y_r][h^{b_{k-2}y_r}, [y_r, b_{k-2}]], \qquad \dots\dots\dots(7)$$

and if $q+1 < k-2$, we may apply Corollary 5.3 to obtain

$$\underline{g} = [h, y_r, b_{q+1}, \dots, b_{k-2}],$$

$$= [h, b_{q+1}, \dots, b_{k-2}, y_r][h^{b_{q+1}y_r}, [y_r, b_{q+1}], b_{q+2}, \dots, b_{k-2}]$$

$$\times \prod_{m=q+1}^{k-3} \left[[h, b_{q+1}, \dots, b_m]^{b_{m+1}y_r}, [y_r, b_{m+1}], b_{m+2}, \dots, b_{k-2}\right] \qquad \dots\dots\dots(8)$$

Step 2 : For $m = q+1, \dots, k-2$,

$$[y_r, b_m] = [b_m, y_r]^{-1} = [b_m, y_r]^{p-1}$$

since $b_m = y_j$ for some j in $\{1, \dots, r-1\}$ and so $[b_m, y_r]$ is a product of conjugates of y_j which commute and are each of order p.

Step 3 : $h^{b_{q+1}y_r} = [g, b_1, \dots, b_q]^{b_{q+1}y_r}$,

and if $q+1 < k-2$, then for $m = q+1, \dots, k-3$,

$$[h, b_{q+1}, \dots, b_m]^{b_{m+1}y_r} = [g, b_1, \dots, b_m]^{b_{m+1}y_r}.$$

So for $m = q, \dots, k-3$, which covers both cases above,

$$[g, b_1, \dots, b_m]^{b_{m+1}y_r} = [g^{b_{m+1}y_r}, b_1^{b_{m+1}y_r}, \dots, b_m^{b_{m+1}y_r}].$$

Now for $j = 1, \dots, m$,

$$b_j^{b_{m+1}y_r} = b_j[\,b_j, b_{m+1}y_r\,]\,,$$
$$= b_j[\,b_j, y_r\,][\,b_j, b_{m+1}\,][\,b_j, b_{m+1}, y_r\,]$$

by Lemma 1.1 ii) . If $[\,b_j, b_{m+1}\,]$ is in D_w , then so is $[\,b_j, b_{m+1}, y_r\,]$, and so by Lemma 4.8 , each of these two commutators may be re-expressed as a product of simple commutators at least as long which start with y_w and have other entries from $\{y_{w+1}, \ldots, y_r\}$. Note that $w < r$ since $m+1 \geq q+1$ implies b_{m+1} is in $\{y_1, \ldots, y_{r-1}\}$. Note also that the y_r being shifted has been absorbed in commutators belonging to $\{D_1, \ldots, D_{r-1}\}$.

Step 4 : Using Step 2 substitute $[\,b_m, y_r\,]^{p-1}$ for $[\,y_r, b_m\,]$

for $m = q+1, \ldots, k-2$, and substitute the products obtained in Step 3 for $b_j^{b_{m+1}y_r}$ for $m = q, \ldots, k-3$ and $j = 1, \ldots, m$ in

$$\underline{g} = [\,g, b_1, \ldots, b_{k-2}, y_r\,] \times$$
$$\prod_{m=q}^{k-3}\left[g^{b_{m+1}y_r}, b_1^{b_{m+1}y_r}, \ldots, b_m^{b_{m+1}y_r}, [y_r, b_{m+1}], b_{m+2}, \ldots, b_{k-2}\right].$$

Note each resulting commutator in the product is of length at least k . Now apply Lemma 1.1 ii) repeatedly to each commutator in the product, $m = q, \ldots, k-3$, to re-express

$$\left[\,g^{b_{m+1}y_r}, b_1^{b_{m+1}y_r}, \ldots, b_m^{b_{m+1}y_r}, [y_r, b_{m+1}], b_{m+2}, \ldots, b_{k-2}\,\right]$$

as a product of commutators of length at least k of the form

$$[\,g^{b_{m+1}y_r}, b_1, \ldots, b_u, {}_{\ell}y_r\,] \qquad \ldots\ldots.(9)$$

where $0 \leq \ell \leq 1$; $b_1, \dots, b_u$ are simple commutators which start with y_j for some j in $\{1, \dots, r\}$ and have other entries from $\{y_{j+1}, \dots, y_r\}$, and such that at most $k_r - 1$ of $b_1, \dots, b_u$ start with y_r . Thus we have re-expressed $\underline{g}$ as a product of commutators of type (6) as required.

Now suppose for induction we have reached the stage $i-1, k_{i-1}-t$, for some t in $\{0, \dots, k_{i-1}-1\}$. We explain what this means. We have already passed the stage $i, 1$ at which we re-expressed $\underline{g}$ as a product $\underline{g}'$ of commutators of type (5) , and k_{i-1} is the maximum of the number of b_j's starting with y_{i-1} in each commutator of type (5) in $\underline{g}'$. Note that this number, k_{i-1} , may exceed the number of y_{i-1}'s in $\underline{g}$ because of the application of Lemma 1.1 ii) , but it is certainly finite since $Q_{n,r}$ is nilpotent. Furthermore, we have shifted the last t of the b_j's starting with y_{i-1} in each commutator of type (5) in $\underline{g}'$, so we have re-expressed $\underline{g}$ as a product $\underline{g}''$ of commutators, each of length at least k , of type (6) .

Those commutators of type (6) in $\underline{g}''$ for which no b'_j in $\{b'_1, \dots, b'_{u'}\}$ starts with y_{i-1} are already of type (5) for $i-1$, so for these commutators stage $i-1, k_{i-1}-t$ is already trivially completed. Let

$$[\, g^z, b_1, \dots, b_u, b_{i-1,s}, \dots, b_{i-1,\ell_{i-1}}, \dots, b_{r-1,\ell_{r-1}}, {}_{\ell}y_r \,]$$

be a commutator of type (6) in $\underline{g}''$ for which there exist b_j's in $\{b_1, \dots, b_u\}$ which start with y_{i-1} . Note if $i-1 < r$ then $\ell = \ell_r$. Let b be the last b_j starting with y_{i-1} . If

$b = b_u$ then the stage $i-1, k_{i-1} - t$ is again trivially completed, so suppose now $b \neq b_u$. Let

$$f = [g^z, b_1, \ldots, b_q, b, b_{q+1}, \ldots, b_u],$$

and $h = [g^z, b_1, \ldots, b_q]$,

where clearly h and q are not necessarily the same as for the case r, k_r above; $b_1, \ldots, b_q$ are simple commutators which start with y_j for some j in $\{1, \ldots, i-1\}$ and other entries from $\{y_{j+1}, \ldots, y_r\}$; and $b_{q+1}, \ldots, b_u$ are simple commutators which start with y_j for some j in $\{1, \ldots, i-2\}$ and have other entries from $\{y_{j+1}, \ldots, y_r\}$.

<u>Step 1</u> : Note that $h \in (C_{p^n})^{(p^r)}$. If $q+1 = u$, by Lemma 5.1,

$$f = [h, b, b_u],$$

$$= [h, b_u, b][h^{b_u b}, [b, b_u]], \qquad \ldots\ldots.(10)$$

and if $q+1 < u$, then by Corollary 5.3,

$$f = [h, b, b_{q+1}, \ldots, b_u],$$

$$= [h, b_{q+1}, \ldots, b_u, b][h^{b_{q+1}b}, [b, b_{q+1}], b_{q+2}, \ldots, b_u] \times$$

$$\prod_{m=q+1}^{u-1} \Big[[h, b_{q+1}, \ldots, b_m]^{b_{m+1}b}, [b, b_{m+1}], b_{m+2}, \ldots, b_u \Big].$$
$$\ldots\ldots.(11)$$

<u>Step 2</u> : For m in $\{q+1, \ldots, u\}$, if b_m is in D_w, i.e. b_m is a simple commutator starting with y_w with other entries from $\{y_{w+1}, \ldots, y_r\}$, then $[b, b_m]$ may be re-expressed by Lemma 4.8 as a product of simple commutators each of length

at least as great as the sum of the lengths of b and b_m, and for which each of the commutators starts with y_w and has other entries from $\{y_{w+1}, \dots, y_r\}$. Note that by choice of b, $w \leq i-2$.

<u>Step 3</u> : $h^{b_{q+1}b} = [g^z, b_1, \dots, b_q]^{b_{q+1}b}$,

and if $q+1 < u$, for $m = q+1, \dots, u-1$,

$$[h, b_{q+1}, \dots, b_m]^{b_{m+1}b} = [g^z, b_1, \dots, b_m]^{b_{m+1}b}.$$

Thus each case above, for $m = q, \dots, u-1$, is given by

$$[g^z, b_1, \dots, b_m]^{b_{m+1}b} = [g^{z\, b_{m+1}b}, b_1^{b_{m+1}b}, \dots, b_m^{b_{m+1}b}].$$

Now $b_j^{b_{m+1}b} = b_j[b_j, b_{m+1}b]$,

$$= b_j[b_j, b][b_j, b_{m+1}][b_j, b_{m+1}, b], \quad j = 1, \dots, m,$$

by Lemma 1.1 ii). If b_j starts with y_{i-1}, then $b, b_j \in D_{i-1}$, which is abelian, and so $[b_j, b] = 1$. Now since $m \geq q$, b_{m+1} is in D_w for some $w < i-1$. Hence $[b_j, b_{m+1}]$ and $[b_j, b_{m+1}, b]$ both belong to D_v for some $v \leq w \leq i-1$, and each can be re-expressed, by Lemma 4.8, as a product of simple commutators starting with y_v and other entries from $\{y_{v+1}, \dots, y_r\}$, each of length at least the sum of the lengths of b_j, b_{m+1} and b_j, b_{m+1}, b respectively. We obtain a similar product for $[b_j, b]$ if $j < i-1$. Note that b has been absorbed by commutators in $\{D_1, \dots, D_{i-2}\}$, and if $j = i-1$ then b_j is the only commutator in the product obtained for $b_j^{b_{m+1}b}$ which starts with y_{i-1}.

<u>Step 4</u> : Substitute the products obtained in Steps 2 and 3 in the

commutators of type (6) in the product $\underline{g}''$ and expand each resulting commutator using Lemma 1.1 ii) repeatedly to obtain the required product of commutators of type (6) for $i-1, k_{i-1}-t$.

The result now follows by induction. □

5.6 LEMMA

Let $\underline{g} \neq 1$ be an element in $\gamma_k(Q_{n,r}) \cap D \setminus \gamma_{k+1}(Q_{n,r}) \cap D$ where $D = (C_{p^n})^{(p^r)}$, the base group of $Q_{n,r} = C_{p^n} \mathrm{wr}^{(p^r)} P_r$. Then $\underline{g}$ can be expressed as a product of commutators of the form

$$[f_1, {}_{k_1}z_1, \dots, {}_{k_r}z_r] \qquad \dots\dots.(12)$$

where $\langle f_1 \rangle = (C_{p^n})_1$; $z_1, \dots, z_r$ are as defined in Definition 4.6 ; $k_i \geq 0$ for $i = 1, \dots, r$, and $1 + \sum_{i=1}^{r} k_i p^{r-i} \geq k$, i.e. (12) is a complex commutator of length at least k .

Before we give the proof, note that a stronger result, which we give later as Proposition 5.7 , can be deduced from results in Chapter III , but these results use knowledge of the nilpotency class of $Q_{n,r}$, which is precisely what we want to find using Lemma 5.6 .

Proof of Lemma 5.6

Note for λ in (p^r) we define f_λ to be the conjugate of f_1

in $(C_{p^n})_\lambda$. By Lemma 2.6 i) , $g \in [D, {}_{k-1}Q_{n,r}]$, and by Lemma 3.17 iii) , $[D, {}_{k-1}Q_{n,r}] = [D, {}_{k-1}P_r]$. Hence g can be expressed as a product of simple commutators each of length k and of the form

$$\left[\prod_{\lambda=1}^{p^r} f_\lambda^{t_\lambda}, b_1, \ldots, b_{k-1}\right] \qquad \ldots\ldots.(13)$$

where $b_1, \ldots, b_{k-1} \in P_r$ and $0 \leq t_\lambda < p^n$ for $\lambda \in (p^r)$.

Rewrite each $b_i \in \{b_1, \ldots, b_{k-1}\}$ as a product of elements from the generating set $\{y_1, \ldots, y_r\}$ of P_r , and expand the resulting commutators (13) using Lemma 1.1 ii) to obtain an expression for g as a product of commutators of length at least k of the form

$$\left[\prod_{\lambda=1}^{p^r} f_\lambda^{t_\lambda}, b_1, \ldots, b_\ell\right]$$

where for λ in (p^r) , t_λ , $b_1, \ldots, b_\ell$ are not necessarily the same as in (13) , $0 \leq t_\lambda < p^n$, and each b_i in $\{b_1, \ldots, b_\ell\}$ belongs to the set $\{y_1, \ldots, y_r\}$. By Lemma 3.17 iii) , since the f_λ's commute,

$$\left[\prod_{\lambda=1}^{p^r} f_\lambda^{t_\lambda}, b_1, \ldots, b_\ell\right] = \prod_{\lambda=1}^{p^r} \{[f_\lambda, b_1, \ldots, b_\ell]^{t_\lambda}\} .$$

Then g is a product of commutators of the form

$$h = [f_\lambda, b_1, \ldots, b_\ell]$$

where $\ell \geq k-1$ and $b_1, \ldots, b_\ell \in \{y_1, \ldots, y_r\}$.

Now take h as the commutator under consideration in Lemma 5.4 . Then h is a product of commutators each of length at least k

and of the form

$$\underline{h} = [\, f_{\lambda'} , b_{1,1} , \ldots , b_{1,\ell_1}, b_{2,1} , \ldots , b_{r-1,\ell_{r-1}}, {}_{\ell_r}y_r \,] \qquad \ldots\ldots(14)$$

and we will show that we can re-express $\underline{h}$ as

$$\underline{h} = \left[\, f_{\lambda'} , z_1^{s_{1,1}} , \ldots , z_1^{s_{1,\ell_1}} , z_2^{s_{2,1}} , \ldots , z_{r-1}^{s_{r-1,\ell_{r-1}}} , {}_{\ell_r}y_r \,\right] \quad \ldots(15)$$

where for i in $\{1, \ldots, r-1\}$ and j in $\{1, \ldots, \ell_i\}$, $0 < s_{i,j} < p$, and if $\ell_i = 0$ then there are no terms in z_i.

Let $\lambda' = (a_0, a_1, \ldots, a_{r-1})$, as given in (3) of Chapter I.

For $i = 2, \ldots, r$, let $c_i = y_i^{a_{i-1}} y_{i+1}^{a_i} \ldots y_r^{a_{r-1}}$, so

$$(\lambda')y_{i-1}^{c_i} = (\lambda')c_i^{-1} y_{i-1} c_i ,$$

$$= \begin{cases} (\nu(a_0+1), a_1, \ldots, a_{r-1}) & \text{if } i=2, \\ (a_0, \ldots, a_{i-3}, \tau(a_{i-2}+1), a_{i-1}, \ldots, a_{r-1}) & \text{if } i>2, \end{cases}$$

i.e. $y_{i-1}^{c_i}$ is the unique conjugate of y_{i-1} with respect to $\langle y_i, \ldots, y_r \rangle$ which acts non-trivially on λ' - see pp. 17-20 of Chapter I.

For $i = 1, \ldots, r-1$, let $\theta_i = \sigma(y_i^{c_{i+1}})$, i.e.

$$\theta_i = \{ (u_0, u_1, \ldots, u_{i-1}, a_i, \ldots, a_{r-1}) : 1 \leq u_0 \leq p, \; 0 \leq u_1, \ldots, u_{i-1} \leq p-1 \}.$$

Note that if $j < i$ then $\theta_j \subset \theta_i$. $\qquad \ldots\ldots\ldots(16)$

We now prove (15) by induction on m, where m indexes the position of the terms in $\underline{h}$: for example, $f_{\lambda'}$ is the first term in $\underline{h}$, for which $m = 1$. We show that on completing the m th stage, we will have re-expressed $\underline{h}$ either as (15), in which case we will have finished, or as

$$\left[f_{\lambda'}, z_1^{s_{1,1}}, \ldots, z, b_{i,j}, \ldots, b_{r-1,\ell_{r-1}}, \ell_r y_r \right] \qquad \ldots\ldots.(17)$$

where $b_{i,j}$ is the $(m+1)$-th term in $\underline{h}$, and

$$z = \begin{cases} z_i^{s_{i,j-1}} & \text{if } j > 1 ; \\ z_w^{s_{w,\ell_w}} & \text{if } j = 0 \text{, where } w \text{ is the greatest integer} \\ & \text{less than } i \text{ for which } \ell_w \neq 0 \text{, i.e. the } m\text{th} \\ & \text{term in } \underline{h} \text{ is } b_{w,\ell_w} . \end{cases}$$

Furthermore, we will show $\sigma([f_{\lambda'}, z_1^{s_{1,1}}, \ldots, z]) \subseteq \theta_i$.
$\ldots\ldots.(18)$

Note we do not need to consider the values of m for which the m th term is a y_r , since we will have already re-expressed $\underline{h}$ as (15) .

The base case of the induction is

$\underline{\underline{m=2}}$: If the second entry is y_r there is nothing to prove, so suppose the second entry is $b_{v,1}$ for some v in $\{1, \ldots, r-1\}$. Now $b_{v,1}$ is in D_v , and so $b_{v,1}$ is a product of conjugates of y_v with respect to $\langle y_{v+1}, \ldots, y_r \rangle$. By definition of c_{v+1} , $y_v^{c_{v+1}}$ is the only one of these conjugates which acts non-trivially on λ' . Then if the power of $y_v^{c_{v+1}}$ in $b_{v,1}$ is $(y_v^{c_{v+1}})^u$, for some u such that $0 < u < p$,

$$\begin{aligned} [f_{\lambda'}, b_{v,1}] &= f_{\lambda'}^{-1} f_{\lambda'}^{b_{v,1}} , \\ &= f_{\lambda'}^{-1} f_{(\lambda')b_{v,1}} , \\ &= f_{\lambda'}^{-1} f_{(\lambda')(y_v^{c_{v+1}})^u} , \end{aligned}$$

$$f_{\lambda'}^{-1} f_{(\lambda')(y_v^{c_{v+1}})^u} = f_{\lambda'}^{-1} f_{\lambda'}^{(y_v^{c_{v+1}})^u} ,$$

$$= [f_{\lambda'} , (y_v^{c_{v+1}})^u] .$$

Since z_v is of maximal length in $\langle y_v, \dots , y_r \rangle \cong \mathbf{P}_{r-v+1}$, by Definition 4.6 and Theorem 3.3 , z_v belongs to the centre of $\langle y_v, \dots , y_r \rangle$. By Corollary 2.33 ,

$$Z(\langle y_v, \dots , y_r \rangle) = \langle \Pi \{ y_v^x : x = y_{v+1}^{k_{v+1}} \dots y_r^{k_r} , \text{ where } k_{v+1}, \dots , k_r \in \{0, \dots , p-1\}\}\rangle \quad \dots\dots\dots(19)$$

and so $z_v = (\prod_x y_v^x)^t$ for some t such that $0 < t < p$.

Hence since y_v^x acts trivially on λ' for $x \neq c_{v+1}$,

$$[f_{\lambda'}, z_v] = [f_{\lambda'} , (y_v^{c_{v+1}})^t] .$$

Now choose $s_{v,1}$ such that $0 < s_{v,1} < p$, $u \equiv t s_{v,1} \bmod p$.

Then

$$[f_{\lambda'} , b_{v,1}] = [f_{\lambda'} , (y_v^{c_{v+1}})^u] ,$$

$$= [f_{\lambda'} , z_v^{s_{v,1}}] ,$$

as required for (17) . Furthermore, $\sigma([f_{\lambda'} , z_v^{s_{v,1}}]) \subseteq \theta_v$,

$$\subseteq \theta_i ,$$

by (16) , where the third entry in $\underline{h}$ is $b_{i,j}$, as required for (18) .

Now suppose we have completed the m th stage. If the $(m+1)$ th term in $\underline{h}$ is y_r , we have finished, so suppose the $(m+1)$ th term in $\underline{h}$ is $b_{i,j}$. By hypothesis,

$$\sigma ([f_{\lambda'} , z_1^{s_{1,1}} , \dots , z]) \subseteq \theta_i .$$

Since $b_{i,j}$ is in D_i, $b_{i,j}$ is a product of conjugates of y_i with respect to $\langle y_{i+1}, \dots, y_r \rangle$, and the only one of these conjugates which acts non-trivially on θ_i is $y_i^{c_{i+1}}$. Let the power of $y_i^{c_{i+1}}$ in $b_{i,j}$ be $(y_i^{c_{i+1}})^u$. Then for $\lambda \in \theta_i$,

$$[f_\lambda, b_{i,j}] = [f_\lambda, (y_i^{c_{i+1}})^u],$$

as above, for some u such that $0 < u < p$, but not necessarily the same u as for $m=2$. By (19),

$$z_i = \left(\Pi\{ y_i^x : x = y_{i+1}^{k_{i+1}} \dots y_r^{k_r} \text{ for } k_{i+1}, \dots, k_r \text{ in } \{0, \dots, p-1\}\}\right)^t$$

for some t such that $0 < t < p$, not necessarily the same t as for $m=2$. Then since y_i^x acts trivially on θ_i for $x \neq c_{i+1}$,

$$[f_\lambda, z_i] = [f_\lambda, (y_i^{c_{i+1}})^t] \qquad \text{for } \lambda \in \theta_i .$$

Choose $s_{i,j}$ such that $0 < s_{i,j} < p$, $u \equiv t s_{i,j} \mod p$. Then for $\lambda \in \theta_i$,

$$[f_\lambda, b_{i,j}] = [f_\lambda, (y_i^{c_{i+1}})^u] = [f_\lambda, z_i^{s_{i,j}}].$$

But $[f_{\lambda'}, z_1^{s_{1,1}}, \dots, z] = \Pi\{ f_\lambda^{t'_\lambda} : \lambda \in \theta_i, 0 \leq t'_\lambda \}$, and so since $(C_{p^n})^{(p^r)}$ is abelian, by Lemma 3.17 iii),

$$[f_{\lambda'}, z_1^{s_{1,1}}, \dots, z, b_{i,j}] = [f_{\lambda'}, z_1^{s_{1,1}}, \dots, z, z_i^{s_{i,j}}]$$

as required for (17). Since $\sigma([f_{\lambda'}, z_1^{s_{1,1}}, \dots, z]) \subseteq \theta_i$, and $\sigma(y_i^{c_{i+1}}) = \theta_i$ by definition,

$$\sigma([f_{\lambda'}, z_1^{s_{1,1}}, \dots, z, z_i^{s_{i,j}}]) \subseteq \theta_i \subseteq \theta_{i'}, \qquad \text{by (16)}$$

where the entry after $b_{i,j}$ in $\underline{h}$ is $b_{i',j'}$ for some j' (- if the entry after $b_{i,j}$ is y_r we have finished -) and we have (18) .

The result (15) now follows by induction . Note again that by definition $y_r = z_r$. Finally, expand (15) using Lemma 1.1 ii) on each of the powers of the z_i's to obtain a new expression for $\underline{h}$ as a product of commutators of the form

$$[f_{\lambda'}, {}_{m_1} z_1, {}_{m_2} z_2, \ldots, {}_{m_{r-1}} z_{r-1}, {}_{\ell_r} z_r]$$

where for $i = 1, \ldots, r\text{-}1$, $m_i \geq 0$ and $k \leq 1 + \sum_{i=1}^{r-1} p^{r-i} m_i + \ell_r$, since z_i is a commutator of length p^{r-i} in $\langle y_i, \ldots, y_r \rangle$ by definition.

It now follows that $\underline{g}$ is a product of commutators of the form

$$[f_\lambda, {}_{m_1} z_1, \ldots, {}_{m_r} z_r]$$

where f_λ is the conjugate of f_1 in $(C_{p^n})_\lambda$, $\lambda \in (p^r)$; and for $i = 1, \ldots, r$, $m_i \geq 0$; and $1 + \sum_{i=1}^{r} m_i p^{r-i} \geq k$, i.e. of type (12) <u>except</u> that we may have $\lambda \neq 1$. The transformation of f_λ's into f_1's is fairly straightforward:

Note that if $\lambda = (a_0, \ldots, a_{r-1})$ then

$$\lambda = (1, 0, \ldots, 0) y_1^{a_0 - 1} y_2^{a_1} \ldots y_r^{a_{r-1}} ,$$

$$= (1, 0, \ldots, 0) z_1^{a_0 - 1} z_2^{a_1} \ldots z_r^{a_{r-1}} .$$

Let $z = z_1^{a_0 - 1} z_2^{a_1} \ldots z_r^{a_{r-1}}$. Recall that by (19) , for $i = 1, \ldots, r$, $\langle z_i \rangle = Z(\langle y_i, \ldots, y_r \rangle)$, and so the z_i's commute. Now

$$[f_1, {}_{m_1} z_1, \ldots, {}_{m_r} z_r]^z$$

$$= [f_1^z, {}_{m_1} z_1^z, \ldots, {}_{m_r} z_r^z],$$

$$= [f_{(1)z}, {}_{m_1} z_1, \ldots, {}_{m_r} z_r] \quad \text{since the } z_i\text{'s commute,}$$

$$= [f_\lambda, {}_{m_1} z_1, \ldots, {}_{m_r} z_r].$$

Hence $[f_\lambda, {}_{m_1} z_1, \ldots, {}_{m_r} z_r]$

$$= [f_1, {}_{m_1} z_1, \ldots, {}_{m_r} z_r][f_1, {}_{m_1} z_1, \ldots, {}_{m_r} z_r, z]. \quad \ldots\ldots.(20)$$

Rewrite z as $z_1^{a_0-1} z_2^{a_1} \ldots z_r^{a_{r-1}}$ in (20) and expand using Lemma 1.1 ii) to a product of commutators each of which starts with $[f_1, {}_{m_1} z_1, \ldots, {}_{m_r} z_r]$ and has other entries from $\{z_1, \ldots, z_r\}$. By Corollary 5.2, since the z_i's commute, we may re-arrange each of the resulting commutators in the form

$$[f_1, {}_{k_1} z_1, \ldots, {}_{k_r} z_r]$$

where for $i = 1, \ldots, r$, $k_i \geqslant m_i$. The result (12) now follows. $\square$

We now come to the alternative proof for the nilpotency class of $Q_{n,r}$.

<u>3.3 THEOREM</u>

The nilpotency class of $C_{p^n} wr^{(p^r)} P_r$ is $p^{r-1}\{p + (p-1)(n-1)\}$.

<u>Proof 2</u>

By Corollary 3.20, there exists a non-trivial simple commutator g of length $c(Q_{n,r})$ which starts with f_1, and has other entries from P_r . By Lemma 5.6 , $\underline{g}$ can be re-expressed as a product of commutators of the form

$$h = [f_1, {}_{k_1} z_1, \ldots, {}_{k_r} z_r]$$

where for $i = 1, \ldots, r$, $k_i \geq 0$ and $1 + \sum_{i=1}^{r} k_i p^{r-i} \geq c(Q_{n,r})$.

Each of the commutators h is in the centre of $Q_{n,r}$, and so by Corollary 2.33 ,

$$h = \left(\prod_{\lambda=1}^{p^r} f_\lambda\right)^t \qquad \text{for some } t \geq 0 .$$

But if $\lambda = (a_0, \ldots, a_{r-1})$ then

$$\lambda = (1, 0, \ldots, 0)z$$

where $z = z_1^{a_0-1} z_2^{a_1} \ldots z_r^{a_{r-1}}$, and so $f_\lambda = f_1^z$. Thus if $h \neq 1$ then there must be at least $(p-1)$ z_i's for each i in $\{1, \ldots, r\}$, i.e. $k_i \geq p-1$.

We also have that h , regarded as a simple commutator in $f_1, z_1, \ldots, z_r$, is in the subgroup $C_{p^n} wr^{(p^r)} \{\langle z_1\rangle \times \langle z_2\rangle \times \ldots \times \langle z_r\rangle\}$ of $Q_{n,r}$. Now $Z = \langle z_1\rangle \times \langle z_2\rangle \times \ldots \times \langle z_r\rangle$ is transitive on (p^r) and of order p^r. Thus by Lemma 2.12 , the transitive permutational representation "(p^r) , Z" is induced by the identity subgroup, so we may regard (p^r) as Z , where we identify $(1)\underline{z}$ in (p^r) with $\underline{z}$ in Z . Thus $C_{p^n} wr^{(p^r)} Z = C_{p^n}$ wr Z .

If $h \neq 1$ our problem is now to find $k_1, \ldots, k_r \geq p-1$ to maximise the length of h in $Q_{n,r}$, where we regard z_i as a commutator of length p^{r-i} for $i = 1, \ldots, r$, given that h is also a simple commutator in $f_1, z_1, \ldots, z_r$ in C_{p^n} wr Z . This is achieved by taking the non-trivial commutator of maximal length in C_{p^n} wr Z given by Corollary 2.23 ,

$$[f_1, {}_{n(p-1)}z_1, {}_{p-1}z_2, \dots, {}_{p-1}z_r],$$

which as a complex commutator of $Q_{n,r}$ is of length

$$1 + n(p-1)p^{r-1} + (p-1)(p^{r-2} + p^{r-3} + \dots + 1),$$

$$= 1 + (n-1)(p-1)p^{r-1} + (p-1)\sum_{i=1}^{r} p^{r-i},$$

$$= 1 + (n-1)(p-1)p^{r-1} + p^r - 1,$$

$$= p^{r-1}\{p + (p-1)(n-1)\} \quad \text{as required.} \qquad \square$$

5.7 PROPOSITION

For $i = 1, \dots, c(Q_{n,r})$, $\gamma_i(Q_{n,r}) \cap D$ is generated modulo $\gamma_{i+1}(Q_{n,r}) \cap D$ by the commutators $[f_1, {}_{k_1}z_1, \dots, {}_{k_r}z_r]$ where $\langle f_1 \rangle = (C_{p^n})_1$; $z_1, \dots, z_r$ are as defined in Definition 4.6 ; D is the base group $(C_{p^n})^{(p^r)}$ of $Q_{n,r}$; and $k_1, \dots, k_r \geq 0$ are such that

$$i = 1 + \sum_{j=1}^{r} k_j p^{r-j},$$

and if w is the smallest integer such that $k_w \neq 0$, then $k_w \leq n(p-1)$, and for $j = w+1, \dots, r$, $k_j \leq p-1$.

Proof

As a complex commutator in $Q_{n,r}$, $[f_1, {}_{k_1}z_1, \dots, {}_{k_r}z_r]$ has length $1 + \sum_{j=1}^{r} k_j p^{r-j}$, and so is in $\gamma_i(Q_{n,r})$. In particular,

$\underline{h} = [f_1, {}_{n(p-1)}z_w, {}_{p-1}z_{w+1}, \dots, {}_{p-1}z_r]$ has length

$$1 + n(p-1)p^{r-w} + (p-1)p^{r-w-1} + (p-1)p^{r-w-2} + \dots + (p-1),$$

$$= 1 + n(p-1)p^{r-w} + p^{r-w} - 1,$$

$$= p^{r-w}\{1 + n(p-1)\},$$

$$= p^{r-w}\{p + (p-1)(n-1)\}.$$

Now $\underline{h} \in \langle f_1, z_w, \dots, z_r \rangle \leq \langle f_1, y_w, \dots, y_r \rangle \cong Q_{n,r-w+1}$, and by Theorem 3.3, $Q_{n,r-w+1}$ has nilpotency class $p^{r-w}\{p + (p-1)(n-1)\}$. But by Corollary 2.23, $\underline{h} \neq 1$. Hence

$$\underline{h} \in \gamma_{c(Q_{n,r-w+1})}(Q_{n,r}) \setminus \gamma_{c(Q_{n,r-w+1})+1}(Q_{n,r}).$$

Since the z_j's commute, by Corollary 5.2,

$$[[f_1, {}_{k_w}z_w, \dots, {}_{k_r}z_r], {}_{n(p-1)-k_w}z_w, {}_{p-1-k_{w+1}}z_{w+1}, \dots, {}_{p-1-k_r}z_r],$$

$$= [f_1, {}_{n(p-1)}z_w, {}_{p-1}z_{w+1}, \dots, {}_{p-1}z_r],$$

$$= \underline{h},$$

and so $[f_1, {}_{k_w}z_w, \dots, {}_{k_r}z_r] \in \gamma_i(Q_{n,r}) \setminus \gamma_{i+1}(Q_{n,r})$.

As in the proof of Lemma 5.6, $[f_1, {}_{k_w}z_w] = [f_1, {}_{k_w}y_w^t]$, $0 < t < p$. So since $z_w, \dots, z_r$ belong to $\langle y_w, \dots, y_r \rangle$ by definition, $[f_1, {}_{k_w}z_w, \dots, {}_{k_r}z_r]$ is a product of conjugates of $[f_1, {}_{k_w}y_w^t]$ with respect to $\langle y_{w+1}, \dots, y_r \rangle$. Hence $[f_1, {}_{k_w}z_w, \dots, {}_{k_r}z_r]$ is in $\langle X_w \rangle$, where X_w is as defined in Definition 4.15.

Hence $[f_1, {}_{k_w}z_w, \dots, {}_{k_r}z_r] \in \langle x_{w,i} \rangle \{\gamma_{i+1}(Q_{n,r}) \cap \langle X_w \rangle\}$, since $x_{w,i}$ generates $\gamma_i(Q_{n,r}) \cap \langle X_w \rangle$ modulo $\gamma_{i+1}(Q_{n,r}) \cap \langle X_w \rangle$. Furthermore, since $[\gamma_i(Q_{n,r}) \cap \langle X_w \rangle : \gamma_{i+1}(Q_{n,r}) \cap \langle X_w \rangle] = p$, by

Theorem 4.23 , it follows that $[f_1, {}_{k_w}z_w, \ldots, {}_{k_r}z_r]$ generates $\gamma_i(Q_{n,r}) \cap \langle X_w \rangle$ modulo $\gamma_{i+1}(Q_{n,r}) \cap \langle X_w \rangle$, and the result now follows from Theorem 4.23 . $\square$

CHAPTER VI : The nilpotency class of $C_{p^n} wr^{(p^2)} B$, where "(p^2), B" is a faithful transitive pair.

Throughout this chapter "(p^2) , B" is a faithful transitive pair.

Note that if B has exponent p^2, then by Corollary 3.14 , the nilpotency class of $C_{p^n} wr^{(p^2)} B$ is $p\{p+(p-1)(n-1)\} = c(Q_{n,2})$. Thus from now on we need only consider groups B of exponent p . The purpose of this chapter is to prove

6.1 THEOREM

Let "(p^2) , B" be a faithful transitive pair such that B has order p^t and is of exponent p . Then the nilpotency class of $C_{p^n} wr^{(p^2)} B$ is $p+n(p-1)(t-1)$.

An important result towards Theorem 6.1 is that transitive subgroups of P_2 of exponent p are of maximal class : I am greatly indebted to Dr. Peter Neumann for pointing this out to me. The proof shown to me used module theory, but we will give a group-theoretic proof below.

6.2 THEOREM

Let B be a transitive subgroup of P_2 of exponent p and order p^t . Then B is of maximal class, and

$$B = \langle \gamma_{p-t+2}(P_2),\ g y_2 \rangle \qquad \text{for some } g \text{ in } \gamma_2(P_2) ,$$

$$= \langle [y_1,\ _{p-t+1} y_2],\ g y_2 \rangle .$$

Proof

The base group $\underline{D}$ of $P_2 = C_p \text{ wr } C_p$ is generated by y_1 and its conjugates, whose supports are mutually disjoint sets. Thus the base group of P_2 is not transitive on (p^2) , and so

$$B = \langle B \cap \underline{D},\ g_1y_2,\ g_2y_2,\ \dots,\ g_ky_2 \rangle$$

for some $k \geq 1$, and $g_1, \dots, g_k$ in $\underline{D}$.

Now for $i = 2, \dots, k$, $g_1y_2(g_iy_2)^{-1} = g_1 g_i^{-1} \in B \cap \underline{D}$, and so $g_iy_2 \in \langle B \cap \underline{D},\ g_1y_2 \rangle$. Thus

$$B = \langle B \cap \underline{D},\ gy_2 \rangle \quad \text{for some } g \text{ in } \underline{D} .$$

Recall that by Theorem 3.3 , $c(P_2) = p$, which is maximal since the order of $P_2 = C_p \text{ wr } C_p$ is p^{p+1} . Thus the lower and upper central series of P_2 coincide - this also follows from Theorem 4.9 - , and for $i = 2, \dots, p$,

$$[\gamma_i(P_2) : \gamma_{i+1}(P_2)] = [Z_{p-i+1}(P_2) : Z_{p-i}(P_2)] = p .$$

The commutator structure of P_2 arises because y_2 does not in general commute with elements of $\underline{D}$. That B is of maximal class arises basically because it contains an element gy_2 , where g is in $\underline{D}$, which does not in general commute with elements in $B \cap \underline{D}$ - except in the trivial case for which B is of order p^2 and thus automatically of maximal class 1 .

We proceed by induction to show that for $i = 1, \dots, t-1$,

$$Z_i(B) \cap \underline{D} = Z_i(P_2) .$$

Note that for a finite p-group G , $Z_j(G) > Z_{j-1}(G)$ for $j = 1, \dots, c(G)$, and $Z_1(G) \neq \langle 1 \rangle$. Note also that by Lemma 3.17 ii) , for f and g in $\underline{D} = (C_p)^{(p)}$, $[f, gy_2] = [f, y_2]$.

<u>i=1</u> : $B \leq C_p \text{ wr } C_p$, so by Lemma 3.17 ii) ,

$$f \in Z(B) \cap \underline{D} \Rightarrow [f, gy_2] = 1 \Rightarrow [f, y_2] = 1 \Rightarrow f \in Z(P_2) .$$

So $Z(B) \cap \underline{D} \leqslant Z(P_2)$. But $Z(P_2)$ is of order p , and so

$$Z(B) \cap \underline{D} = Z(P_2) \qquad \text{as required.}$$

Now suppose for induction that $Z_i(B) \cap \underline{D} = Z_i(P_2)$ for some $i \leqslant t-2$. Then for f in $Z_{i+1}(B) \cap \underline{D}$,

$$[f, gy_2] \in Z_i(B) \cap \underline{D} = Z_i(P_2) \qquad \text{by hypothesis,}$$

$$\Rightarrow [f, y_2] \in Z_i(P_2) \qquad \text{by Lemma 3.17 ii) ,}$$

$$\Rightarrow f \in Z_{i+1}(P_2) .$$

Hence $Z_{i+1}(B) \cap \underline{D} \leqslant Z_{i+1}(P_2)$. But $[Z_{i+1}(P_2) : Z_i(P_2)] = p$ and so we have the result by induction. In particular,

$$Z_{t-1}(B) \cap \underline{D} = Z_{t-1}(P_2) > Z_{t-2}(P_2) = Z_{t-2}(B) \cap \underline{D} ,$$

and so B has class at least $t-1$. Since B has order p^t , it follows that B is of maximal class $t-1$, and

$$B \cap \underline{D} = Z_{t-1}(B) \cap \underline{D} = Z_{t-1}(P_2) = \gamma_{p-t+2}(P_2) .$$

Hence $B = \langle \gamma_{p-t+2}(P_2), gy_2 \rangle$ for some g in $\underline{D}$.

By Lemma 4.8 , since $\gamma_i(P_2) \cap \underline{D} = \gamma_i(P_2)$ for $i = 2, \dots, p$,

$$\gamma_{p-t+2}(P_2) = \langle [y_1, {}_k y_2] : k \in \{p-t+1, \dots, p-1\} \rangle .$$

Now for $k > p-t+1$,

$$[y_1, {}_k y_2] = [y_1, {}_{p-t+1} y_2, {}_{k-(p-t+1)} y_2] ,$$

$$= [y_1, {}_{p-t+1} y_2, {}_{k-(p-t+1)} gy_2]$$

by Lemma 3.17 iii),

$$\in \langle [y_1, {}_{p-t+1} y_2], gy_2 \rangle .$$

Thus $B = \langle [y_1, {}_{p-t+1} y_2], gy_2 \rangle$, for some g in $\underline{D}$.

Finally we need to show that g is in $\gamma_2(P_2)$. This follows immediately from Lemma 6.4 below : recall B is of exponent p .

$\square$

<u>6.3 DEFINITION</u>

We will write $w_0 w_1 \dots w_{p-1} v$ for the element

$$y_1^{w_0} \left(y_1^{y_2}\right)^{w_1} \dots \left(y_1^{y_2^{p-1}}\right)^{w_{p-1}} y_2^{v}$$

of P_2 , where for $i = 0, \dots , p-1$, $0 \leqslant w_i \leqslant p-1$ and $0 \leqslant v \leqslant p-1$. $\square$

To make the following lemmas, Lemma 6.4 and Lemma 6.7 , more comprehensible, we resort to a visual aid.

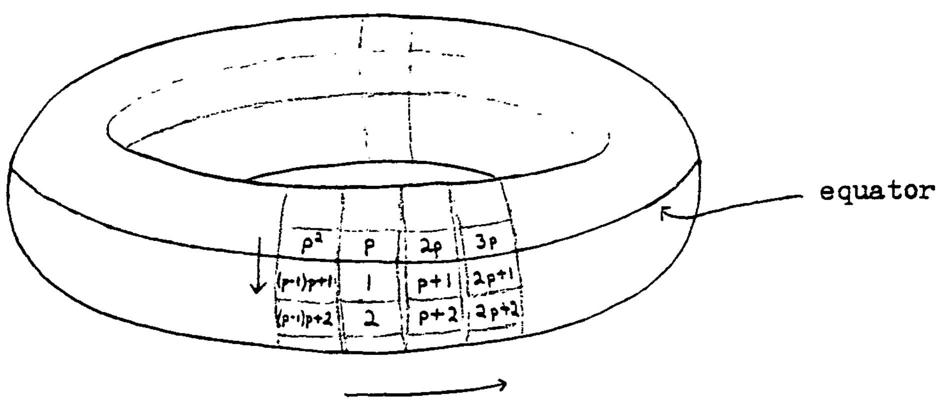


Fig. 12

Imagine a torus made up of p wheels. The rim of each wheel is divided into p equal parts by lines parallel to the equator. The symbols $1, \dots , p$ are placed in successive boxes round the "zeroth" wheel which represents the permutation

$$y_1 = y_1^{y_2^0} = (1, \dots , p) .$$

The symbols $p+1, p+2, \dots, 2p$ are placed in successive boxes round the "first" wheel so that when 1 is aligned with $p+1$, 2 is aligned with $p+2$, and so on. This wheel represents the permutation

$$y_1^{y_2} = (p+1, p+2, \dots , 2p) .$$

In general, the ith wheel represents the permutation

$$y_1^{y_2^i} = (ip+1, ip+2, \dots , (i+1)p) ,$$

with these symbols placed in successive boxes round the wheel. Then, if we align $1, p+1, 2p+1, \dots , (p-1)p+1$, the permutation y_2 is represented by the p cycles

$$(j, p+j, 2p+j, \dots , (p-1)p + j) \quad \text{for} \quad j = 1, \dots , p$$

parallel to the equator.

Then, when the wheels are aligned to represent y_2, if $1 \leqslant a_0 \leqslant p$ and $0 \leqslant a_1 \leqslant p-1$, the symbol $a_0 + a_1 p$ is to be found on the a_1th wheel on the latitude corresponding to a_0. To obtain the symbol $(a_0 + a_1 p) y_2^k$ for some $k \geqslant 1$, we find $a_0 + a_1 p$ and count round k boxes on the a_0 latitude from $a_0 + a_1 p$ to $a_0 + \tau(a_1 + 1) p$ to $a_0 + \tau(a_1 + 2) p$ and so on, to obtain

$$(a_0 + a_1 p) y_2^k = a_0 + \tau(a_1 + k) p ,$$

as required by (7) of Chapter I. Clearly, when $k = p$ we get

$$\begin{aligned} (a_0 + a_1 p) y_2^p &= a_0 + \tau(a_1 + p) p , \\ &= a_0 + a_1 p , \end{aligned}$$

as required since $y_2^p = 1$. If $0 \leqslant a \leqslant p-1$, to obtain say $(a_0 + a_1 p)\left(y_1^{y_2^{a}}\right)^k$ we first check if $a = a_1$. If not then $y_1^{y_2^{a}}$ acts as the identity on $a_0 + a_1 p$ since $\sigma(y_1^{y_2^{a}}) = \{1 + ap, 2 + ap, \ldots, (a+1)p\}$. If $a = a_1$ then we find $a_0 + a_1 p$ and count round k successive boxes on the a_1 th wheel, from $a_0 + a_1 p$ to $\nu(a_0+1) + a_1 p$ to $\nu(a_0+2) + a_1 p$ and so on to obtain

$$(a_0 + a_1 p)\left(y_1^{y_2^{a_1}}\right)^k = \nu(a_0+k) + a_1 p .$$

Again,

$$\begin{aligned}(a_0 + a_1 p)\left(y_1^{y_2^{a_1}}\right)^p &= \nu(a_0+p) + a_1 p ,\\ &= a_0 + a_1 p ,\end{aligned}$$

as required since $y_1^{y_2^{a_1}}$ has the same order as y_1 , namely p .

Now suppose we wish to find $(a_0 + a_1 p)(g y_2^k)$ where $k \geqslant 1$, $g = w_0 w_1 \ldots w_{p-1} 0$, and $\sum_{i=0}^{p-1} w_i \equiv 0 \mod p$. This can be done very simply on the model as we now describe, and the justification for this is the proof of the lemma below : in other words, we give a visual interpretation of the lemma here.

We first re-align the p wheels, so that at the equator we have

$$1, \nu(1+w_0) + p, \nu(1+w_0+w_1) + 2p, \ldots, \nu(1 + \sum_{j=0}^{p-2} w_j) + (p-1)p$$

on the 0 th, 1 st, ..., $(p-1)$ th wheels respectively. Then $g y_2$ is represented by the p cycles parallel to the equator

$$(\ell, \nu(\ell+w_0) + p, \ldots, \nu(\ell + \sum_{j=0}^{p-2} w_j) + (p-1)p)$$

for $\ell = 1, \ldots, p$. Now proceed as described above for y_2^k to find $(a_0 + a_1 p)(gy_2)^k$: we find $a_0 + a_1 p$ on the model and count round k boxes, from $a_0 + a_1 p$ to

$$\nu(a_0 + w_{a_1}) + \tau(a_1 + 1)p = \nu(a_0 + w_{\tau(a_1)}) + \tau(a_1 + 1)p$$

to

$$\nu(a_0 + w_{\tau(a_1)} + w_{\tau(a_1 + 1)}) + \tau(a_1 + 2)p$$

and so on to obtain

$$(a_0 + a_1 p)(gy_2)^k = \nu(a_0 + \sum_{j=a_1}^{a_1+k-1} w_{\tau(j)}) + \tau(a_1 + k)p .$$

Note that from the lemma below, Lemma 6.4 , gy_2 is of order p if $\sum_{i=0}^{p-1} w_i \equiv 0 \mod p$, and from above,

$$(a_0 + a_1 p)(gy_2)^p = \nu(a_0 + \sum_{j=a_1}^{a_1+p-1} w_{\tau(j)}) + \tau(a_1 + p)p ,$$

$$= a_0 + a_1 p$$

as required, since $\sum_{i=0}^{p-1} w_i \equiv 0 \mod p$.

We could have obtained $(a_0 + a_1 p)(gy_2)^k$ from the model when it was aligned to represent y_2 : recall that

$$g = y_1^{w_0} \left(y_1^{y_2}\right)^{w_1} \cdots \left(y_1^{y_2^{p-1}}\right)^{w_{p-1}} ,$$

so combine operating with elements of the form $\left(y_1^{y_2^a}\right)^r$ with operations by y_2 on the model to obtain the result. This method is the way we prove Lemma 6.4 .

6.4 LEMMA

Let g be an element of P_2 such that $g = w_0 w_1 \dots w_{p-1} 0$, where for $i = 0, \dots, p-1$, $0 \leqslant w_i \leqslant p-1$. Then for $k \geqslant 1$, $1 \leqslant a_0 \leqslant p$, $0 \leqslant a_1 \leqslant p-1$,

$$(a_0 + a_1 p)(gy_2)^k = \nu(a_0 + \sum_{j=a_1}^{a_1+k-1} w_{\tau(j)}) + \tau(a_1+k)\, p \,.$$

Hence $|gy_2| = p$ if and only if $\sum_{j=0}^{p-1} w_j \equiv 0 \mod p$, if and only if

$g \in Y_2(P_2)$.

<u>Proof</u>

Recall $y_1^{y_2^k} = (k'p+1, k'p+2, \ldots, (k'+1)p)$ for $0 \leq k' \leq p-1$,

and $k' \equiv k \mod p$. For $1 \leq a_0 \leq p$, $0 \leq a_1 \leq p-1$, and $k \geq 1$,

$$(a_0 + a_1 p)\, y_2^k = a_0 + \tau(a_1+k)\, p \,.$$

We proceed by induction on k .

<u>k=1</u> : Since $a_0 + a_1 p \in \{ a_1 p + 1, \ldots, \tau(a_1+1)\, p \} = \sigma(y_1^{y_2^{a_1}})$,

$$\begin{aligned}(a_0 + a_1 p)\, g y_2 &= (\nu(a_0 + w_{a_1}) + a_1 p)\, y_2 \,, \\ &= \nu(a_0 + w_{a_1}) + \tau(a_1+1)\, p \,, \\ &= \nu(a_0 + w_{\tau(a_1)}) + \tau(a_1+1)\, p \,, \text{ as required.}\end{aligned}$$

Suppose the result is true for some k in $\mathbb{Z}^+$. Then

$$(a_0 + a_1 p)(gy_2)^{k+1} = (\nu(a_0 + \sum_{j=a_1}^{a_1+k-1} w_{\tau(j)}) + \tau(a_1+k)\, p)\, g y_2$$

by the induction hypothesis,

$$= (\nu(a_0 + \sum_{j=a_1}^{a_1+k-1} w_{\tau(j)} + w_{\tau(a_1+k)}) + \tau(a_1+k)p)y_2$$

$$= \nu(a_0 + \sum_{j=a_1}^{a_1+k} w_{\tau(j)}) + \tau(a_1+k+1)\, p \,,$$

as required for the induction.

Now $\nu(a_0 + \sum_{j=a_1}^{a_1+k-1} w_{\tau(j)}) + \tau(a_1+k)p = a_0 + a_1 p$ if and only if

$k \equiv 0 \mod p$ and $\sum_{j=a_1}^{a_1+k-1} w_{\tau(j)} \equiv 0 \mod p$.

Thus $(a_0 + a_1 p)(gy_2)^p = a_0 + a_1 p$ if and only if

$\sum_{j=a_1}^{a_1+p-1} w_{\tau(j)} = \sum_{j=0}^{p-1} w_j \equiv 0 \mod p$, i.e. $|gy_2| = p$ if and only

if $\sum_{j=0}^{p-1} w_j \equiv 0 \mod p$, if and only if $g \in \gamma_2(P_2)$ by Lemma 4.16 .

$\square$

6.5 DEFINITION

The pairs "Λ , B" and "Θ , C" are said to be isomorphic if there exists an isomorphism $\theta : B \longrightarrow C$ and a bijection $\phi : \Lambda \longrightarrow \Theta$ such that for b in B and λ in Λ ,

$$(\lambda b)\phi = (\lambda\phi)(b\theta) . \qquad \square$$

We have the general theorem

6.6 THEOREM

Let the pairs "Λ , B" and "Θ , C" be isomorphic, and let A be a group. Then $A \operatorname{wr}^{\Lambda} B \cong A \operatorname{wr}^{\Theta} C$.

Proof

Let $\theta : B \longrightarrow C$ be the isomorphism of groups, and let $\phi : \Lambda \longrightarrow \Theta$ be the bijection such that for b in B and λ in Λ , $(\lambda b)\phi = (\lambda\phi)(b\theta)$. Define $\theta' : A^{\Lambda} \longrightarrow A^{\Theta}$ co-ordinate-wise by $(f_\lambda)\theta' = f_{\lambda\phi}$, where $f_\lambda \in A_\lambda$ and $f_{\lambda\phi} \in A_{\lambda\phi}$ both correspond to $f \in A$. Then θ' is clearly well-defined and is an isomorphism : bijection is clear, and if $f, g \in A$,

$$(f_\lambda g_\lambda)\theta' = ((fg)_\lambda)\theta' = (fg)_{\lambda\phi} = f_{\lambda\phi} g_{\lambda\phi} ,$$

where $f_\lambda, g_\lambda \in A_\lambda$ and $f_{\lambda\phi}, g_{\lambda\phi} \in A_{\lambda\phi}$ correspond to f and g in A, and so θ' is a homomorphism.

Now define $\psi : A \,\mathrm{wr}^\Lambda B \longrightarrow A \,\mathrm{wr}^\Theta C$ by $(fb)\psi = (f\theta')(b\theta)$, for f in A^Λ and b in B. Then ψ is clearly well-defined. We show ψ is a homomorphism. For f,g in A^Λ and b_1, b_2 in B,

$$\begin{aligned}(fb_1\, gb_2)\psi &= (f g^{b_1^{-1}} b_1 b_2)\psi ,\\ &= (f g^{b_1^{-1}})\theta' (b_1 b_2)\theta ,\\ &= f\theta' \, (g^{b_1^{-1}})\theta' \; b_1\theta \; b_2\theta ;\end{aligned}$$

and

$$\begin{aligned}(fb_1)\psi (gb_2)\psi &= f\theta' \; b_1\theta \; g\theta' \; b_2\theta ,\\ &= f\theta' \; (g\theta')^{(b_1\theta)^{-1}} \; b_1\theta \; b_2\theta ;\end{aligned}$$

so we need to check $(g^{b_1^{-1}})\theta' = (g\theta')^{(b_1\theta)^{-1}}$.

Now $g = \prod_{\lambda\in\Lambda} h_\lambda$ for $h_\lambda \in A_\lambda$, $\lambda \in \Lambda$. Thus

$$(g^{b_1^{-1}})\theta' = \prod_{\lambda\in\Lambda} (h_\lambda^{b_1^{-1}})\theta' = \prod_{\lambda\in\Lambda} (h_{\lambda b_1^{-1}}\theta') = \prod_{\lambda\in\Lambda} (h_{(\lambda b_1^{-1})\phi}) ;$$

and

$$\begin{aligned}(g\theta')^{(b_1\theta)^{-1}} &= \Big(\prod_{\lambda\in\Lambda} (h_\lambda\theta') \Big)^{b_1^{-1}\theta} ,\\ &= \Big(\prod_{\lambda\in\Lambda} (h_{\lambda\phi})\Big)^{b_1^{-1}\theta} ,\\ &= \prod_{\lambda\in\Lambda} (h_{\lambda\phi}^{\,b_1^{-1}\theta}) ,\\ &= \prod_{\lambda\in\Lambda} (h_{(\lambda\phi)(b_1^{-1}\theta)}) ,\\ &= \prod_{\lambda\in\Lambda} (h_{(\lambda b_1^{-1})\phi}) \quad \text{as required.}\end{aligned}$$

Since θ and θ' are isomorphisms, ψ is clearly bijective, and so ψ is an isomorphism. $\square$

6.7 LEMMA

Let g be an element in $\gamma_2(P_2)$. Then for $i = 1, \ldots, p-1$,

$$"(p^2) , \langle [y_1, {}_iy_2], gy_2 \rangle" \cong "(p^2) , \langle [y_1, {}_iy_2], y_2 \rangle" .$$

Proof

Let $\underset{\sim}{D}$ be the base group $(C_p)^{(p)}$ of $P_2 = C_p \text{ wr } C_p$, let $G = \langle [y_1, {}_iy_2], gy_2 \rangle$ and let $B = \langle [y_1, {}_iy_2], y_2 \rangle$.

Since $\underset{\sim}{D}$ is abelian, for f in $\underset{\sim}{D}$, $f^{gy_2} = f^{y_2}$ and so

$$\begin{aligned} G \cap \underset{\sim}{D} &= \langle [y_1, {}_iy_2]^y : y \in \langle gy_2 \rangle \rangle , \\ &= \langle [y_1, {}_iy_2]^{y'} : y' \in \langle y_2 \rangle \rangle , \\ &= B \cap \underset{\sim}{D} . \end{aligned}$$

Define the map $\theta : B \longrightarrow G$ by $(fy_2^k)\theta = f(gy_2)^k$ for $k \geq 0$, and $f \in B \cap \underset{\sim}{D} = G \cap \underset{\sim}{D}$. Since gy_2 is of order p by Lemma 6.4 , the map θ is clearly well-defined. We also have θ is a homomorphism, for if f and h are in $B \cap \underset{\sim}{D}$ and $k_1, k_2 \geq 0$,

$$\begin{aligned} (fy_2^{k_1} hy_2^{k_2})\theta &= (fh^{y_2^{-k_1}} y_2^{k_1+k_2})\theta , \\ &= fh^{y_2^{-k_1}} (gy_2)^{k_1+k_2} , \\ &= fh^{(gy_2)^{-k_1}} (gy_2)^{k_1+k_2} , \\ &= f(gy_2)^{k_1} h(gy_2)^{k_2} , \\ &= (fy_2^{k_1})\theta (hy_2^{k_2})\theta . \end{aligned}$$

Clearly for f in $B \cap \underline{D}$ and $k \geq 0$, $(fy_2^k)\theta = 1$ if and only if $f = 1$ and $k \equiv 0 \bmod p$, i.e. $f = 1$ and $y_2^k = 1$, so θ is injective, and since G and B are finite it follows that θ is an isomorphism.

Let $g = w_0 \dots w_{p-1} 0$. Define a bijection $\phi : (p^2) \longrightarrow (p^2)$ as follows. For $1 \leq a_0 \leq p$, $a_0\phi = a_0$. For $1 \leq a_0 \leq p$ and $1 \leq a_1 \leq p-1$,

$$(a_0 + a_1 p)\phi = \nu(a_0 + \sum_{j=0}^{a_1-1} w_j) + a_1 p .$$

The map ϕ is a bijection since $\{ \nu(a_0 + \sum_{j=0}^{a_1-1} w_j) : a_0 = 1, \dots, p \}$ is the set (p).

Note that $a_0 + \tau(a_1+v)p = (a_0 + a_1 p) y_2^v$, and

$$\nu(a_0 + \sum_{j=0}^{a_1+v-1} w_{\pi(j)}) + \tau(a_1+v)p = (a_0 + \tau(a_1+v)p)\phi .$$

We need to show that for $\lambda \in (p^2)$ and $b \in B$, $(\lambda b)\phi = (\lambda\phi)(b\theta)$. Let $\lambda = a_0 + a_1 p$ where $1 \leq a_0 \leq p$ and $0 \leq a_1 \leq p-1$, and let $x_{i+1} = [y_1, {}_i y_2]$. Then any element b in B is of the form

$$\begin{aligned} b &= x_{i+1}^{u_0} \left(x_{i+1}^{y_2}\right)^{u_1} \dots \left(x_{i+1}^{y_2^{p-1}}\right)^{u_{p-1}} y_2^v = f y_2^v \text{, say ,} \\ &= x_{i+1}^{u_0} \left(x_{i+1}^{g y_2}\right)^{u_1} \dots \left(x_{i+1}^{(g y_2)^{p-1}}\right)^{u_{p-1}} y_2^v , \end{aligned}$$

where for $j = 0, \dots, p-1$, $0 \leq u_j \leq p-1$, and $0 \leq v \leq p-1$. Note $f \in B \cap \underline{D} = G \cap \underline{D}$. Now $(a_0 + a_1 p) f = \nu(a_0 + u) + a_1 p$ where $\left(y_1^{y_2^{a_1}}\right)^u$ is the power of $y_1^{y_2^{a_1}}$ in f, and so

$$((a_0 + a_1 p) f)\phi = (\nu(a_0 + u) + a_1 p)\phi ,$$

$$(\nu(a_o+u)+a_1p)\phi = \begin{cases} \nu(a_o+u+\sum_{j=o}^{a_1-1} w_j)+a_1p & \text{if } a_1 \neq 0, \\ \nu(a_o+u) & \text{if } a_1 = 0; \end{cases}$$

$$= \begin{cases} (\nu(a_o+\sum_{j=o}^{a_1-1} w_j)+a_1p)\,f & \text{if } a_1 \neq 0, \\ a_o f & \text{if } a_1 = 0; \end{cases}$$

$$= ((a_o+a_1p)\phi)\,f \;;$$

i.e. ϕ and elements of $B\cap \underline{D} = G\cap \underline{D}$ commute.

$$(a_o+a_1p)\phi(b)\theta$$

$$= (a_o+a_1p)\phi\, f(gy_2)^v ,$$

$$= \begin{cases} (\nu(a_o+\sum_{j=o}^{a_1-1} w_j)+a_1p)\,(gy_2)^v\, f^{(gy_2)^v} & \text{if } a_1 \neq 0, \\ a_o\,(gy_2)^v\, f^{(gy_2)^v} & \text{if } a_1 = 0, \end{cases}$$

$$= \begin{cases} (\nu(a_o+\sum_{j=o}^{a_1+v-1} w_{\tau(j)})+\tau(a_1+v)\,p)\,f^{y_2^v} & \text{if } a_1 \neq 0, \text{ by Lemma 6.4}, \\ (\nu(a_o+\sum_{j=o}^{v-1} w_{\tau(j)})+\tau(v)\,p)\,f^{y_2^v} & \text{if } a_1 = 0, \text{ by Lemma 6.4}, \end{cases}$$

$$= (\nu(a_o+\sum_{j=o}^{a_1+v-1} w_{\tau(j)})+\tau(a_1+v)\,p)\,f^{y_2^v} ,$$

$$= (a_o+\tau(a_1+v)\,p)\phi\, f^{y_2^v} ,$$

$$= (a_o+a_1p)\,y_2^v\,\phi\, f^{y_2^v} ,$$

$$= (a_o+a_1p)\,y_2^v\, f^{y_2^v}\,\phi \quad \text{since } \phi \text{ and } f^{y_2^v} \in B\cap \underline{D} = G\cap \underline{D} \text{ commute},$$

$$= ((a_o+a_1p)b)\phi$$ as required, and we have the result. □

From Theorem 6.2 , Theorem 6.6 and Lemma 6.7 we have immediately

6.8 THEOREM

Let B be a transitive subgroup of P_2 of exponent p and order p^t . Then

$$C_{p^n} \, \mathrm{wr}^{(p^2)} B \cong C_{p^n} \, \mathrm{wr}^{(p^2)} \langle \gamma_{p-t+2}(P_2) \, , \, y_2 \rangle \; . \qquad \square$$

The proof of Theorem 6.1 now proceeds in much the same way as the second proof of Theorem 3.3 given in Chapter V . We first need a couple of definitions.

6.9 DEFINITION

For $t = 2, \ldots , p$ let $B_t = \langle \gamma_{p-t+2}(P_2), y_2 \rangle$. $\square$

6.10 DEFINITION

For $i = 2, \ldots , p$ let $x_i = [y_1, {}_{i-1}y_2]$. $\square$

6.11 REMARK

Note that $x_i = z_{1,i}$ where $z_{1,i}$ is as in Definition 4.7 , and that by Theorem 6.2 , $B_t = \langle x_{p-t+2}, y_2 \rangle$. Also, for $0 \leqslant s \leqslant t-2$, $x_{p-t+2+s} = [x_{p-t+2}, {}_s y_2]$, a commutator of length $1 + s$ in B_t . $\square$

Analogous to Lemma 5.4 we have

6.12 LEMMA

Let $\underline{g} \neq 1$ be a simple commutator of length k in $C_{p^n} wr^{(p^2)} B_t$ with first entry g from $D = (C_{p^n})^{(p^2)}$ and other entries from $\{ x_{p-t+2}, y_2 \}$. Then $\underline{g}$ can be re-expressed as a product of commutators each of which, when regarded as a complex commutator in $C_{p^n} wr^{(p^2)} B_t$, is of length at least k, and of the form

$$[g^z, b_1, \ldots, b_\ell, {}_m y_2] \qquad \ldots\ldots.(1)$$

where g^z is a conjugate of g, $\ell \geq 0$, $m \geq 0$ and $b_1, \ldots, b_\ell \in \{ x_{p-t+2}, x_{p-t+3}, \ldots, x_p \}$.

Proof

The proof of this lemma is obtained by substituting x_{p-t+2} for y_1 in the proof of Lemma 5.4, with $r = 2$. □

Analogous to Lemma 5.6 we have

6.13 LEMMA

Let $\underline{g} \neq 1$ be an element in

$$\gamma_k(C_{p^n} wr^{(p^2)} B_t) \cap D \setminus \gamma_{k+1}(C_{p^n} wr^{(p^2)} B_t) \cap D ,$$

where $D = (C_{p^n})^{(p^2)}$, the base group of $C_{p^n} wr^{(p^2)} B_t$. Then $\underline{g}$ can be expressed as a product of commutators of the form

$$[f_1, {}_{k_1} z_1, {}_{k_2} z_2] = [f_1, {}_{k_1} x_p, {}_{k_2} y_2]$$

where $\langle f_1 \rangle = (C_{p^n})_1$; z_1 and z_2 are as in Definition 4.6 ; $k_i \geq 0$ for $i = 1, 2$, and $1 + k_1(t-1) + k_2 \geq k$.

Proof

Note that $[f_1, {}_{k_1} z_1, {}_{k_2} z_2]$ is a complex commutator in $C_{p^n} wr^{(p^2)} B_t$ of length $1 + k_1(t-1) + k_2$ since $z_1 = x_p$ is a simple commutator of length $t-1 = c(B_t)$ in B_t , and $z_2 = y_2$ is of length 1 in B_t .

By Lemma 2.6 i) , $\underline{g} \in [D, {}_{k-1} C_{p^n} \mathrm{wr}^{(p^2)} B_t]$,

$= [D, {}_{k-1} B_t]$ by Lemma 3.17 iii) , since D is abelian.

Now $B_t = \langle x_{p-t+2}, y_2 \rangle$ by Remark 6.11 , so $\underline{g}$ can be expressed as a product of commutators each of length at least k of the form

$$[g, c_1, \dots, c_{k'}] \qquad \dots\dots\dots(2)$$

where $c_1, \dots, c_{k'}$ are products of x_{p-t+2} and y_2, and g is in D . Expand each commutator (2) using Lemma 1.1 ii) repeatedly, to obtain a new expression for $\underline{g}$ as the product of commutators of the form

$$h = [g, c'_1, \dots, c'_{k''}]$$

where $k'' \geq k-1$, and $c'_1, \dots, c'_{k''} \in \{x_{p-t+2}, y_2\}$.

The proof now proceeds exactly as for Lemma 5.6 , except we substitute x_{p-t+2} for y_1 in the h of the proof of Lemma 5.6, and apply Lemma 6.12 rather than Lemma 5.4 . □

<u>Proof of Theorem 6.1</u>

This is a minor modification of Proof 2 of Theorem 3.3 given in Chapter V : substitute $C_{p^n} \mathrm{wr}^{(p^2)} B_t$ for $Q_{n,r}$, 2 for r , and thus (p^2) for (p^r), B_t for P_r , Lemma 6.13 for Lemma 5.6 , $z = z_1^{a_0-1} z_2^{a_1}$ for $z = z_1^{a_0-1} z_2^{a_1} \dots z_r^{a_{r-1}}$, $Z = \langle z_1 \rangle \times \langle z_2 \rangle$ for $Z = \langle z_1 \rangle \times \dots \times \langle z_r \rangle$, and we finally have

$[f_1, {}_{n(p-1)}z_1, {}_{p-1}z_2]$ is a complex commutator in $C_{p^n} \mathrm{wr}^{(p^2)} B_t$ of length $c(C_{p^n} \mathrm{wr}^{(p^2)} B_t)$, namely $1 + n(p-1)(t-1) + p-1 = p + n(p-1)(t-1)$ as required. $\square$

6.14 REMARK

Since it is nice to have Theorem 6.8, we have given a proof of Theorem 6.1 using this fact, but it really just simplifies notation, for Lemmas 6.12 and 6.13 generalise for $\langle x_{p-t+2}, gy_2 \rangle$ rather than B_t, where g is in $\gamma_2(P_2)$. This follows since $[x_{p-t+2}, {}_s gy_2] = [x_{p-t+2}, {}_s y_2]$ and z_1 and gy_2 commute as $\langle z_1 \rangle = Z(P_2)$, i.e. $\langle z_1, gy_2 \rangle = \langle z_1 \rangle \times \langle gy_2 \rangle$. In the proof of Theorem 6.1 we could thus just substitute gy_2 for y_2, to obtain the result. $\square$

CHAPTER VII : Towards a formula for the nilpotency class of $A \text{ wr}^{\Lambda} B$ where A is a group and "Λ , B" is a pair.

In this chapter we attempt to put back the scaffolding possibly used by Shield to obtain Theorem 2.2 - Corollary 5.5 [17] . We do this in order to show how a generalisation from the standard wreath product to the permutational wreath product may be made, and in particular, what we would need for Conjecture 3.2 to be true. We do not aim to always give formal arguments for the standard case.

For easy reference we generalise the notion of a simple cpp-commutator.

7.1 DEFINITION

A simple pseudo-commutator of a group G is an expression of the form

$$[[\cdots[g_1^{t_1}, g_2]^{t_2}, \ldots , g_{\ell-1}]^{t_{\ell-1}}, g_\ell]^{t_\ell}$$

where for $i = 1, \ldots , \ell$, $g_i \in G$ and $t_i \in \mathbb{Z}$. If G is periodic we may choose t_i in $\mathbb{N}$. □

Note that simple commutators are simple pseudo-commutators.

7.2 DEFINITION

Let p be a fixed prime, and let a and b be integers such that $a \geq b \geq 0$, $a \geq 1$. Let

$$\underline{g} = [[\cdots[g_1^{t_1}, g_2]^{t_2}, \ldots , g_{\ell-1}]^{t_{\ell-1}}, g_\ell]^{t_\ell}$$

be a non-trivial simple pseudo-commutator in the group G such that for $i = 1, \ldots, \ell$, $t_i = u_i p^{v_i}$ where p^{v_i} is the highest power of p dividing t_i . Note that since $\underline{g} \neq 1$, $t_i \neq 0$ for $i = 1, \ldots, \ell$. Then we define the <u>(a,b,1)-length</u> of $\underline{g}$ to be

$$a + b v_1 + a + b v_2 + \cdots + a + b v_{\ell-1} + a + b v_\ell$$

$$= a\ell + b \sum_{i=1}^{\ell} v_i \ . \qquad \square$$

Clearly if $\underline{g}$ has (a,b,1)-length w then $\underline{g} \in \gamma_w^{a,b,1}(G)$: see p.24 of Chapter I for the definition of the (a,b,1)-series.

We now look at the standard wreath product. Let A be a nilpotent p-group of finite exponent, and let B be a finite p-group. Then by Theorem 2.1 , $A \text{ wr } B$ is nilpotent, and by Lemma 3.19 and a result such as Lemma 2.6 i), there exists a simple commutator $\underline{g}_c \neq 1$ in $[A^B, {}_{c-1} A \text{ wr } B]$ where $c = c(A \text{ wr } B)$.

Let $1 \neq \underline{g} = [g_1, g_2 b_2, \ldots, g_j b_j]$, where $1 \leq j \leq c$, and for $i = 1, \ldots, j$, $g_i \in A^B$ and $b_i \in B$. We also have for $i = 1, \ldots, j$,

$$g_i = \prod_{b \in B} x_{i,b} \qquad \text{where} \quad x_{i,b} \in A_b \ .$$

Then by applying Lemma 1.1 i), ii) repeatedly to $\underline{g}$ we can re-express $\underline{g}$ as a product of non-trivial simple commutators each of length at least j and of the form

$$1 \neq \underline{g}' = [h_{1,b_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,b_2}, c_{2,1}, \ldots, c_{2,k_2}, h_{3,b_3}, \ldots, h_{\ell,b_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}] \qquad \ldots\ldots\ldots(1)$$

where for $i = 1, \ldots, \ell$, $k_i \geq 0$, $b_i \in B$, the element h_{i,b_i} of

A_{b_i} corresponds to the element h_i of A , and $c_{i,1}, \ldots, c_{i,k_i}$ are in B .

In particular, since g_c is in $\gamma_c(A \text{ wr } B)$, g_c is a product of non-trivial simple commutators of type (1) of length exactly c , and let g_c' be one such commutator. We want to obtain an expression for each k_i in g_c' given that we know

$$\sum_{i=1}^{\ell} k_i + \ell = c \ .$$

We will show that a non-trivial simple commutator g' of type (1) can be re-expressed as a non-trivial simple pseudo-commutator

$$g' = [\cdots[\cdots[h_{1,b_\ell}^{t_1}, h_{2,b_\ell}]^{t_2}, \ldots, h_{\ell-1,b_\ell}]^{t_{\ell-1}}, h_{\ell,b_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}] \qquad \ldots\ldots\ldots(2)$$

where for $i = 1, \ldots, \ell-1$ we have $t_i > 0$ and $h_{i,b_\ell} \in A_{b_\ell}$ corresponds to the element h_i in A . Note that since we have "removed" $c_{i,1}, \ldots, c_{i,k_i}$ for $i = 1, \ldots, \ell-1$, (2) may have shorter (1,0,1)-length, i.e. shorter nilpotency length, than (1), but we will see we can "put back" at least as many terms as we took out.

For $i = 1, \ldots, \ell$, let

$$f_{i,b_i} = [h_{1,b_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,b_2}, \ldots, h_{i,b_i}] ,$$

the commutator g' of (1) up to and including h_{i,b_i} . Note that since f_{i,b_i} starts with $h_{1,b_1} \in A^B \lhd A \text{ wr } B$,

$$f = [h_{1,b_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,b_2}, \ldots, c_{i-1,k_{i-1}}] \in A^B ,$$

and so since the co-ordinate subgroups $\{A_b\}_{b\in B}$ commute element-wise, by Lemma 1.1 i),

$$f_{i,b_i} = [f, h_{i,b_i}] \in A_{b_i} .$$

For b in B and $i = 1, \dots, \ell$, let $f_{i,b} = (f_{i,b_i})^{b_i^{-1}b}$, which is an element in A_b.

We can now define the t_i's of (2) : for $i = 1, \dots, \ell-1$, let $f_{i,b_{i+1}}^{t_i}$ be the power of $f_{i,b_{i+1}}$ in the product of conjugates of f_{i,b_i} given by

$$[f_{i,b_i}, c_{i,1}, \dots, c_{i,k_i}] . \qquad \dots\dots\dots(3)$$

Note that any pair of conjugates in (3) commute since if they are not equal then they belong to distinct co-ordinate subgroups of A^B. Then for $i = 1, \dots, \ell-1$,

$$f_{i+1,b_{i+1}} = [f_{i,b_i}, c_{i,1}, \dots, c_{i,k_i}, h_{i+1,b_{i+1}}],$$

$$= [f_{i,b_{i+1}}^{t_i}, h_{i+1,b_{i+1}}] \quad \text{by Lemma 1.1 i)} .$$

$$\dots\dots\dots(4)$$

We now prove by induction that for $i = 1, \dots, \ell$,

$$f_{i,b_i} = [[\cdots[h_{1,b_i}^{t_1}, h_{2,b_i}]^{t_2}, \dots, h_{i-1,b_i}]^{t_{i-1}}, h_{i,b_i}] .$$

$$\dots\dots\dots(5)$$

<u>i=1</u> : $f_{1,b_1} = h_{1,b_1}$ by definition.

Now suppose the result is true for $i-1$. Then by (4) the result is true for i and induction gives us (5) as required. This shows that f_{i,b_i} can be expressed as a simple pseudo-commutator in A_{b_i}. But

$$\underline{g}' = [f_{\ell,b_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}],$$

$$= [\cdots[\cdots[h_{1,b_\ell}^{t_1}, h_{2,b_\ell}]^{t_2}, \ldots, h_{\ell-1,b_\ell}]^{t_{\ell-1}}, h_{\ell,b_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}]$$

by (5) and so $\underline{g}'$ is of the form (2) with t_i defined as above : note that since $\underline{g}' \neq 1$ we must have $t_i \neq 0$ for $i = 1, \ldots, \ell-1$, and so since A wr B has finite exponent, we can choose $t_i > 0$. Also note that since $f_{i,b_i}^{b_i^{-1}} = f_{i,1}$,

$$\underline{g}' = (\underline{g}')^{b_\ell^{-1}} [(\underline{g}')^{b_\ell^{-1}}, b_\ell],$$

$$= [f_{\ell,1}, c_{\ell,1}^{b_\ell^{-1}}, c_{\ell,2}^{b_\ell^{-1}}, \ldots, c_{\ell,k_\ell}^{b_\ell^{-1}}] \times$$

$$[f_{\ell,1}, c_{\ell,1}^{b_\ell^{-1}}, c_{\ell,2}^{b_\ell^{-1}}, \ldots, c_{\ell,k_\ell}^{b_\ell^{-1}}, b_\ell],$$

where each of these two commutators is a simple pseudo-commutator of (1,0,1)-length at least as great as that of $\underline{g}'$, and is of the form

$$\underline{g}'' = [\cdots[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell, \underline{c}_{\ell,1}, \underline{c}_{\ell,2}, \ldots, \underline{c}_{\ell,\underline{k}_\ell}] \quad \ldots\ldots.(6)$$

where $\underline{k}_\ell \geq k_\ell$, and $h_1, \ldots, h_\ell$, $t_1, \ldots, t_{\ell-1}$ are as above for $\underline{g}'$, and $\underline{c}_{\ell,1}, \ldots, \underline{c}_{\ell,\underline{k}_\ell} \in B$.

We now look in general at simple pseudo-commutators in A of the form

$$1 \neq \underline{h} = [[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell] \quad \ldots\ldots.(7)$$

where $\ell, h_1, \ldots, h_\ell$ and $t_1, \ldots, t_{\ell-1}$ are not necessarily the same as in $\underline{g}''$ above. Let $\underline{h}$ be of order p^v, and for

$i = 1, \ldots, \ell-1$, let $t_i = u_i p^{v_i}$ where p^{v_i} is the highest power of p dividing t_i, so $u_i \not\equiv 0 \bmod p$. We claim we may rewrite $\underline{h} \in A = A_1 \leqslant A \text{ wr } B$ as a simple pseudo-commutator in $A \text{ wr } B$,

$$\underline{h} = [\cdots[\cdots[h_{1,b_1'}^{u_1'}, c_{1,1}', \ldots, c_{1,k_1'}', h_{2,b_2'}]^{u_2'}, c_{2,1}', \ldots, c_{2,k_2'}', , h_{3,b_3'}]^{u_3'}, \ldots, c_{\ell-1\, k_{\ell-1}'}', h_\ell] \qquad \ldots\ldots.(8)$$

where for $i = 1, \ldots, \ell-1$, $u_i' \not\equiv 0 \bmod p$, $b_i' \in B$, $h_{i,b_i'} = h_i^{b_i'}$; $k_i' = c(C_{p^{v_{i+1}}} \text{ wr } B) - 1$, and $c_{i,1}', \ldots, c_{i,k_i'}' \in B$. In addition, if $\underline{h} = f_{\ell,1}$, given in (5), we claim that for $i = 1, \ldots, \ell-1$, $k_i' \geqslant k_i$, where k_i is as in (1).

Furthermore, $\underline{h} \in A_1$, since $h_\ell \in A_1$, and is of order p^v, so there exists a non-trivial simple pseudo-commutator in $A \text{ wr } B$ of the form

$$\underline{\underline{h}} = [\underline{h}, c_{\ell,1}', \ldots, c_{\ell,k_\ell'}'] \qquad \ldots\ldots.(9)$$

where $k_\ell' = c(C_{p^v} \text{ wr } B) - 1$, $c_{\ell,1}', \ldots, c_{\ell,k_\ell'}' \in B$, and in addition if $\underline{h} = f_{\ell,1}$ we claim $k_\ell' \geqslant \underline{k}_\ell \geqslant k_\ell$, for which $\underline{k}_\ell$ is given in (6) and k_ℓ is given in (2). We will prove all these claims later in the more general context of the permutational wreath product, in Lemma 7.9.

It follows that given a non-trivial simple pseudo-commutator $\underline{h}$ in A of the form (7) we can obtain a non-trivial simple pseudo-commutator in $A \text{ wr } B$ of $(1,0,1)$-length

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr } B) + c(C_{p^v} \text{ wr } B) \qquad \ldots\ldots.(10)$$

and so for each $\underline{h}$, (10) is a lower bound for $c(A \text{ wr } B)$. In particular, if $\underline{h} = f_{\ell,1}$, we obtain a non-trivial simple pseudo-commutator (9) of (1,0,1)-length at least that of $\underline{g}'$, i.e. of (1,0,1)-length equal to c, and so for $i = 1, \ldots, \ell-1$, the values for k_i in (1) are given by

$$k_i = c(C_{p^{v_{i+1}}} \text{ wr } B) - 1 ,$$

and in (1), $k_\ell = c(C_{p^v} \text{ wr } B) - 1$. Thus

$$c(A \text{ wr } B) = \sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr } B) + c(C_{p^v} \text{ wr } B) .$$

At this stage, from looking at examples of standard wreath products, we might guess that

$$c(C_{p^n} \text{ wr } B) = c(C_p \text{ wr } B) + (p-1)\,d(B)\,(n-1) . \qquad \ldots\ldots.(11)$$

Shield calls the constant $(p-1)d(B)$ "b(B)", but to avoid confusion with elements of B,

7.3 DEFINITION

For a finite p-group B let $\underline{d}(B) = (p-1)\,d(B)$. $\square$

Then if we assume (11) is true, we may simplify (10) as

$$\sum_{i=1}^{\ell-1} \{ c(C_p \text{ wr } B) + \underline{d}(B)\, v_i \} + c(C_p \text{ wr } B) + \underline{d}(B)(v-1) ,$$

$$= c(C_p \text{ wr } B)\ell + \underline{d}(B) \{ \sum_{i=1}^{\ell-1} v_i + v - 1 \} .$$

7.4 DEFINITION

From now on let $a(B) = c(C_p \text{ wr } B)$. $\square$

Note we are not assuming Shield's definition of the constant $a(B)$ of a group B. Note also that by our assumption (11), $a(B) \geq \underline{d}(B)$, since we clearly cannot have $c(C_{p^2} \text{ wr } B)$ greater than $2c(C_p \text{ wr } B) = 2a(B)$.

We now introduce the $(a(B), \underline{d}(B), 1)$-series of A into the argument. The simple pseudo-commutator $\underline{h} \neq 1$ of A given in (7) has $(a(B), \underline{d}(B), 1)$-length $a(B)\ell + \underline{d}(B)\sum_{i=1}^{\ell-1} v_i$ from Definition 7.2, and so the non-trivial pseudo-commutator $\underline{h}^{p^{v-1}}$ has $(a(B), \underline{d}(B), 1)$-length

$$a(B)\ell + \underline{d}(B)\{\sum_{i=1}^{\ell-1} v_i + v - 1\}.$$

If we assume (11) is true, then this is precisely the $(1,0,1)$-length of the non-trivial simple pseudo-commutator $\underline{\underline{h}}$ of (9) which we obtained from $\underline{h}$. Thus the nilpotency class of $A \text{ wr } B$ is bounded above by the $(a(B), \underline{d}(B), 1)$-class of A, which by Corollary 1.4 is

$$\max\{ a(B)w + \underline{d}(B)(s(w) - 1) : 1 \leq w \leq c(A) \} \qquad \ldots\ldots\ldots(12)$$

where a commutator of nilpotency, i.e. $(1,0,1)$-, length w in A has order at most $p^{s(w)}$.

We show that this is also a lower bound by constructing a non-trivial simple pseudo-commutator of $(1,0,1)$-length (12), and thus we obtain (12) as the nilpotency class of $A \text{ wr } B$.

After Lemma 6.7 [16], Shield remarks

" the well-known result that ... the commutators of weight i generating $\gamma_i(G)$ may be restricted to being ... left-normed carries across also to the other series" ,

in other words, Lemma 1.5 here generalises to the (a,b,e)-series. The generalisation to the (a,b,1)-series is presumably

7.5 LEMMA

For a group G , $\gamma_i^{a,b,1}(G)$ is generated modulo $\gamma_{i+1}^{a,b,1}(G)$ by elements of the form g^{p^v} where g is a simple commutator of length u in $\gamma_u(G) \setminus \gamma_{u+1}(G)$ and g^{p^v} has (a,b,1)-weight $au+bv=i$. □

Note that this is a stronger result than Lemma 1.2 .

7.6 COROLLARY

If G is an (a,b,1)-nilpotent group of (a,b,1)-class k , then there exists an element $1 \neq g^{p^t}$ in G such that g is a simple commutator of length u in $\gamma_u(G) \setminus \gamma_{u+1}(G)$ and $au+bt=k$. □

Corollary 7.6 does not follow immediately from Lemma 1.2 , but is used by Shield in the proof of Theorem 5.4 [17] , although in [16] after the above remark Shield claims he does not need Lemma 7.5 in [17] , and so omits the proof. We will, for ease of exposition, assume Lemma 7.5 is true, but note that we could still obtain (12) as our lower bound for $c(A \text{ wr } B)$ without it, by defining and using complex pseudo-commutators, which have the usual problem of complex commutators of being difficult to write down in general form.

Now Corollary 7.6 tells us that there exists a non-trivial simple pseudo-commutator in A of $(a(B), \underline{d}(B), 1)$-length equal to the $(a(B), \underline{d}(B), 1)$-class of A, and so using (8) and (9), and assuming (11), we can obtain a non-trivial simple pseudo-commutator in A wr B of $(1,0,1)$-length equal to the $(a(B), \underline{d}(B), 1)$-class of A. Thus if (11) is true, we have proved

$$c(A \text{ wr } B) = \max\{\, a(B)\, w + \underline{d}(B)(s(w) - 1) : 1 \leq w \leq c(A) \,\}$$

where a commutator of nilpotency weight w in A has order at most $p^{s(w)}$. This last construction is the idea behind Shield's proof of Theorem 5.4 a) [17].

We have now given the part of the argument which we will later generalise to the permutational wreath product. The gap that still needs to be filled here is (11).

We could show $c(C_{p^n} \text{ wr } B) \geq c(C_p \text{ wr } B) + \underline{d}(B)(n-1)$ ....(13)

if we could obtain a non-trivial complex commutator of length $c(C_p \text{ wr } B) = a(B)$ of the form

$$h = [\, f_1, b_1, \ldots, b_m, {}_{p^{v-1}-1}\, g \,] \qquad \text{.......(14)}$$

which is not a pth power, where $\langle f_1 \rangle = (C_{p^n})_1$, $b_1, \ldots, b_m$ are in B, $g \notin \langle b_1, \ldots, b_m \rangle$ but $g \in \gamma_u(B) \setminus \gamma_{u+1}(B)$ and is such that $up^{v-1} = d(B)$: such an element exists by Lemma 1.2.

For if h is not a pth power, neither is

$$[\, f_1, b_1, \ldots, b_m \,] = \Pi\{\, f_b^{u_b} : b \in \langle b_1, \ldots, b_m \rangle \,\}$$

where $f_b = f_1^{\,b}$ and $0 \leq u_b < p^n$, and so there exists b' in $\langle b_1, \ldots, b_m \rangle$ such that $u_{b'}$ is not divisible by p, i.e. $f_{b'}^{u_{b'}}$ is of order p^n. Then by Corollary 2.30, since $|g| = p^v$,

$$[f_{b'}^{u_{b'}}, {}_{q-1} g] \neq 1 \qquad \ldots\ldots\ldots(15)$$

where $q = p^{v-1}\{p + (p-1)(n-1)\}$. Thus

$$[f_1, b_1, \ldots, b_m, {}_{q-1} g]$$

$$= [\Pi\{f_b^{u_b} : b \in \langle b_1, \ldots, b_m \rangle\}, {}_{q-1} g],$$

$$= \Pi\{[f_b^{u_b}, {}_{q-1} g] : b \in \langle b_1, \ldots, b_m \rangle\} \quad \text{by Lemma 3.17 iii) since the conjugates of } f_1 \text{ commute,}$$

$$\neq 1 \quad \text{by (15)},$$

and so we would obtain (13) . Shield constructs a commutator of type (14) in the proof of Theorem 5.4 a) [17] , using the so-called standard basis of B , which is shown to exist in Theorem 2.4 [17] , and gives us in addition a formula for $a(B) = c(C_p \text{ wr } B)$. Shield also uses the standard basis to obtain the important result Lemma 3.8 [17] , which we will see gives us

$$c(C_{p^n} \text{ wr } B) \leq c(C_p \text{ wr } B) + \underline{d}(B)(n-1) .$$

We will interpret what Lemma 3.8 [17] says for the group C_{p^n} wr B . We claim Lemma 3.8 [17] implies that for b in B , if $\langle f_b \rangle = (C_{p^n})_b$, $\ell \geq c(C_p \text{ wr } B) = a(B)$ and $b_1, \ldots, b_\ell \in B$, then

$$[f_b, b_1, \ldots, b_\ell] = g^p$$

where $g \in \{\gamma_{a(B)-\underline{d}(B)+1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B\}((C_{p^n})^B)^p$ . ...(16)

Note $((C_{p^n})^B)^p \cong (C_{p^{n-1}})^B$.

Rewrite each b_i in its standard form relative to the standard basis of B - see p. 59 [17] - and then expand $[f_b, b_1, \ldots, b_\ell]$ using Lemma 1.1 ii) repeatedly to obtain a new expression for $[f_b, b_1, \ldots, b_\ell]$ as a product of commutators each of length at least $\ell+1$ and of the form

$$[f_b, b_1', \ldots, b_{\ell'}'] = f_b^{(b_1'-1)(b_2'-1)\ldots(b_{\ell'}'-1)}$$

where for $i = 1, \ldots, \ell'$, b_i' belongs to the standard basis of B, and f_b is as above.

Now $(b_1'-1)(b_2'-1)\ldots(b_{\ell'}'-1)$ is a <u>monomial</u> in the integer group ring $\mathbb{Z}B$ - defined on p. 69 [17] - , and has <u>weight</u> at least $\ell' \geq \ell$. Lemma 3.8 [17] now says since $\ell' \geq a(B)$, $[f_b, b_1', \ldots, b_{\ell'}']$ may be re-expressed as a product of elements of the form

$$(f_b^{\chi_d})^p$$

where χ_d is a monomial of weight at least $\ell' - \underline{d}(B)$, which is greater than or equal to $a(B) - \underline{d}(B)$ since $\ell' \geq a(B)$. The monomial χ_d is thus of the form

$$\chi_d = (c_1^{p^{r_1}}-1)(c_2^{p^{r_2}}-1)\ldots(c_k^{p^{r_k}}-1)$$

where for $i = 1, \ldots, k$, c_i is in the standard basis of B, $r_i \geq 0$, and if $c_i \in \gamma_{u_i}(B) \setminus \gamma_{u_i+1}(B)$, then the weight of χ_d is given by

$$\sum_{i=1}^{k} u_i p^{r_i} \geq \ell' - \underline{d}(B) \geq a(B) - \underline{d}(B) . \qquad \ldots\ldots.(17)$$

An induction on Lemma 3.6 [17] yields

$$(c_i^{p^{r_i}}-1) = (c_i-1)^{p^{r_i}} + p\psi$$

for some ψ in $\mathbb{Z}B$, and so

$$\chi_d = \left((c_1-1)^{p^{r_1}} + p\psi_1\right)\left((c_2-1)^{p^{r_2}} + p\psi_2\right)\ldots\left((c_k-1)^{p^{r_k}} + p\psi_k\right)$$

$$= (c_1-1)^{p^{r_1}}(c_2-1)^{p^{r_2}}\ldots(c_k-1)^{p^{r_k}} + p\underline{\psi}$$

for some ψ in $\mathbb{Z}B$. But then

$$f_b^{\chi_d} = f_b^{(c_1-1)^{p^{r_1}} \cdots (c_k-1)^{p^{r_k}}} \cdot f_b^{p\psi} ,$$

$$= [f_b, {}_{p^{r_1}} c_1, {}_{p^{r_2}} c_2, \ldots, {}_{p^{r_k}} c_k] f_b^{p\psi} ,$$

$$\in \{ \gamma_{a(B) - \underline{d}(B)+1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B \} ((C_{p^n})^B)^p \quad \text{by } (17),$$

and we obtain the result (16) .

As a corollary to (16) we will obtain

$$c(C_{p^n} \text{ wr } B) \leq c(C_p \text{ wr } B) + \underline{d}(B)(n-1) . \qquad \ldots\ldots.(18)$$

We proceed by induction on m to show that if $1 \leq m \leq n$, $b \in B$, $\langle f_b \rangle = (C_{p^n})_b$, $\ell \geq c(C_p \text{ wr } B) + \underline{d}(B)(m-1)$ and $b_1, \ldots , b_\ell \in B$, then

$$[f_b, b_1, \ldots , b_\ell] = g^{p^m}$$

for some g in $\{ \gamma_{a(B) - \underline{d}(B) + 1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B \} ((C_{p^n})^B)^p$.

$$\ldots\ldots.(19)$$

<u>m=1</u> : This is just (16) .

Suppose now the result (19) is true for m . Then if $\ell \geq c(C_p \text{ wr } B) + \underline{d}(B)\, m$, $b_1, \ldots , b_\ell \in B$, and $k = c(C_p \text{ wr } B) + \underline{d}(B)(m-1)$,

$$[f_b, b_1, \ldots , b_\ell] = [[f_b, b_1, \ldots , b_k], b_{k+1}, \ldots , b_\ell] ,$$

$$= [g^{p^m} , b_{k+1}, \ldots , b_\ell]$$

where $g \in \{ \gamma_{a(B) - \underline{d}(B)+1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B \} ((C_{p^n})^B)^p$ by the induction hypothesis.

Since $g \in (C_{p^n})^B$ which is abelian, by Lemma 3.17 iii),

$$[g^{p^m}, b_{k+1}, \dots , b_\ell] = [g, b_{k+1}, \dots , b_\ell]^{p^m} .$$

By Lemma 2.6 i), and repeated application of Lemma 1.1 ii) and Lemma 3.17 iii), we can re-express g as a product of simple commutators each of length at least $a(B) - \underline{d}(B) + 1$ and of the form $[f_{b'}, c_1, \dots , c_u]$, where $b' \in B$, $\langle f_{b'} \rangle = (C_{p^n})_{b'}$, $u \geq a(B) - \underline{d}(B)$, and $c_1, \dots , c_u \in B$. Then substituting this product for g in $[g, b_{k+1}, \dots , b_\ell]$, and expanding using Lemma 3.17 iii) repeatedly on the product we re-express $[g, b_{k+1}, \dots , b_\ell]$ as a product of simple commutators each of length at least $a(B) - \underline{d}(B) + 1 + \ell - k \geq a(B) - \underline{d}(B) + 1 + \underline{d}(B) = a(B) + 1$ of the form

$$[f_{b'}, c_1, \dots , c_{u'}]$$

where $\langle f_{b'} \rangle = (C_{p^n})_{b'}$, $u' \geq a(B)$, and $c_1, \dots , c_{u'} \in B$. Thus by (16),

$$[f_{b'}, c_1, \dots , c_{u'}] = \underline{g}^p$$

where $\underline{g} \in \{ \gamma_{a(B) - \underline{d}(B) + 1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B \} ((C_{p^n})^B)^p$, and so since $(C_{p^n})^B$ is abelian,

$$[g, b_{k+1}, \dots , b_\ell] = (\underline{g}')^p$$

where $\underline{g}' \in \{ \gamma_{a(B) - \underline{d}(B) + 1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B \} ((C_{p^n})^B)^p$, i.e.

$$[f_b, b_1, \dots , b_\ell] = (\underline{g}')^{p^{m+1}} ,$$

as required for the induction, and we obtain (19).

It now follows from (19) that if $\ell \geq c(C_p \text{ wr } B) + \underline{d}(B)(n-1)$, $\langle f_1 \rangle = (C_{p^n})_1$, and $b_1, \dots , b_\ell \in B$, then

$$[f_1, b_1, \ldots, b_\ell]$$
$$\in (\gamma_{a(B)-\underline{d}(B)+1}(C_{p^n} \text{ wr } B) \cap (C_{p^n})^B)^{p^n} \quad ((C_{p^n})^B)^{p^{n+1}} ,$$
$$= \langle 1 \rangle .$$

Thus since by Corollary 3.20 there exist extra-special commutators of length $c(C_{p^n} \text{ wr } B)$ which are non-trivial, there exist non-trivial extra-special commutators of every length less than or equal to $c(C_{p^n} \text{ wr } B)$, and consequently we have

$$c(C_{p^n} \text{ wr } B) \leqslant c(C_p \text{ wr } B) + \underline{d}(B)(n-1)$$

as required, so we have proved assumption (11) .

We now remove the scaffolding : all the manipulations on simple commutators in A wr B , of the form $[g_1b_1, g_2b_2, \ldots, g_\ell b_\ell]$, are confined to Lemma 4.2 [17] . Note that in fact Lemma 4.2 [17] is a stronger result than we need for $c(A \text{ wr } B)$ - it is also used to prove the result for $d(A \text{ wr } B)$ - and we could make do with just

$$[g_1b_1, \ldots, g_\ell b_\ell] \in [b_1, \ldots, b_\ell]\, \gamma_\ell^{a(B), \underline{d}(B), 1}(A) . \quad \ldots\ldots(20)$$

Note also that the proof of Lemma 4.2 [17] uses Lemma 3.8 [17] . Lemma 4.2 [17] gives us the upper bound for $c(A \text{ wr } B)$, and as mentioned above, Theorem 5.4 a) [17] gives us the lower bound.

We have completed the discussion of the standard wreath product, and will now try to generalise to the permutational wreath product : we aim towards Conjecture 3.2 .

By the remark after Theorem 2.4 , we may restrict our attention to $W = A \text{ wr}^{\Lambda} B$, where A is a non-trivial nilpotent p-group of finite exponent, B is a finite p-group, and "Λ , B" is a faithful transitive pair, i.e. $\Lambda = (p^r)$ for some r .

7.7 LEMMA

For $1 \leqslant w \leqslant c(W)$, every element g in A^Λ of weight w can be expressed as a product of non-trivial simple pseudo-commutators of the form

$$1 \neq h = [\cdots[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell, b_1, \ldots, b_m] \qquad \ldots\ldots.(21)$$

where $\ell \geqslant 1$, $m \geqslant 0$, $h_1, \ldots, h_\ell \in A_1$, i.e. we have identified a symbol in Λ as "1" , $b_1, \ldots, b_m \in B$; and if for $i = 1, \ldots, \ell-1$, $t_i = u_i p^{v_i}$ where p^{v_i} is the highest power of p dividing t_i then

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \operatorname{wr}^\Lambda B) + 1 + m \geqslant w . \qquad \ldots\ldots.(22)$$

Proof

Note since $h \neq 1$, $t_i \neq 0$ for $i = 1, \ldots, \ell-1$, and $u_i \not\equiv 0 \mod p$. Since $A \operatorname{wr}^\Lambda B$ is nilpotent by Theorem 2.4 , it follows from Lemma 2.6 i) that g can be expressed as a product of simple commutators of length at least w of the form

$$\underline{g} = [g_1 \ , g_2 b_2, \ldots, g_j b_j]$$

where $j \geqslant w$, and for $i = 1, \ldots, j$, $g_i \in A^\Lambda$ and $b_i \in B$.
Now for $i = 1, \ldots, j$ we also have

$$g_i = \prod_{\lambda \in \Lambda} x_{i,\lambda} \qquad \text{where } x_{i,\lambda} \in A_\lambda .$$

By applying Lemma 1.1 i), ii) repeatedly to $\underline{g}$ we can re-express

$\underline{g}$ as a product of non-trivial simple commutators each of length at least $j \geqslant w$ and of the form

$$1 \neq \underline{g}' = [h_{1,\lambda_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,\lambda_2}, c_{2,1}, \ldots, c_{2,k_2}, h_{3,\lambda_3}, \ldots, h_{\ell,\lambda_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}] \qquad \ldots\ldots\ldots(23)$$

where $\ell \geqslant 1$, and for $i = 1, \ldots, \ell$, $\lambda_i \in \Lambda$, $h_{i,\lambda_i} \in A_{\lambda_i}$ corresponds to the element h_i of A, $k_i \geqslant 0$, and $c_{i,1}, \ldots, c_{i,k_i} \in B$.

For $i = 1, \ldots, \ell$, let

$$f_{i,\lambda_i} = [h_{1,\lambda_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,\lambda_2}, c_{2,1}, \ldots, h_{i,\lambda_i}],$$

the commutator $\underline{g}'$ up to and including h_{i,λ_i}. Note that since f_{i,λ_i} starts with $h_{1,\lambda_1} \in A^\Lambda \lhd A \operatorname{wr}^\Lambda B$,

$$f = [h_{1,\lambda_1}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,\lambda_2}, \ldots, c_{i-1,k_{i-1}}] \in A^\Lambda,$$

and so since the co-ordinate subgroups $\{A_\lambda\}_{\lambda \in \Lambda}$ commute elementwise, we have by Lemma 1.1 i) that

$$f_{i,\lambda_i} = [f, h_{i,\lambda_i}] \in A_{\lambda_i}.$$

For λ in Λ there exists b in B such that $\lambda_i b = \lambda$ since B is transitive on Λ. Then define $f_{i,\lambda} = (f_{i,\lambda_i})^b$, which is in A_λ.

We can now define the t_i's in (21) : let $f_{i,\lambda_{i+1}}^{t_i}$ be the power of $f_{i,\lambda_{i+1}}$ in the product of conjugates of f_{i,λ_i} given by

$$[f_{i,\lambda_i}, c_{i,1}, \ldots, c_{i,k_i}]. \qquad \ldots\ldots\ldots(24)$$

Any pair of conjugates in (24) commutes, since if they are not equal then they belong to distinct co-ordinate subgroups of A^{Λ} . Then by Lemma 1.1 i) , for $i = 1, \ldots, \ell-1$,

$$f_{i+1,\lambda_{i+1}} = [f_{i,\lambda_i}, c_{i,1}, \ldots, c_{i,k_i}, h_{i+1,\lambda_{i+1}}],$$

$$= [f_{i,\lambda_{i+1}}^{t_i}, h_{i+1,\lambda_{i+1}}]. \qquad \ldots\ldots\ldots(25)$$

We now prove by induction that for $i = 1, \ldots, \ell$,

$$f_{i,\lambda_i} = [[\cdots[h_{1,\lambda_i}^{t_1}, h_{2,\lambda_i}]^{t_2}, \ldots, h_{i-1,\lambda_i}]^{t_{i-1}}, h_{i,\lambda_i}]. \qquad \ldots\ldots\ldots(26)$$

$\underline{\underline{i=1}}$: $f_{1,\lambda_1} = h_{1,\lambda_1}$ by definition.

Now suppose the result is true for $i-1$. Then by (25) the result is true for i , and induction gives us (26) .

This shows that f_{i,λ_i} can be expressed as a simple pseudo-commutator in A_{λ_i} . But

$$\underline{g}' = [f_{\ell,\lambda_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}],$$

$$= [\cdots[\cdots[h_{1,\lambda_\ell}^{t_1}, h_{2,\lambda_\ell}]^{t_2}, \ldots, h_{\ell-1,\lambda_\ell}]^{t_{\ell-1}}, h_{\ell,\lambda_\ell}, c_{\ell,1}, \ldots, c_{\ell,k_\ell}].$$

Since B is transitive on Λ , there exists b in B such that $\lambda_\ell b = 1$. Identify A with the co-ordinate subgroup A_1 of A^{Λ} , so identifying h_i with $h_{i,1}$ for $i = 1, \ldots, \ell$. Then

$$\underline{g}' = [\cdots[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell, c_{\ell,1}^b, c_{\ell,2}^b, \ldots, c_{\ell,k_\ell}^b] \times$$

$$[\cdots[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell, c_{\ell,1}^b, c_{\ell,2}^b, \ldots, c_{\ell,k_\ell}^b, b^{-1}], \qquad \ldots\ldots\ldots(27)$$

which gives us g as a product of commutators of the form (21), but we still need to check (22).

Let $t_i = u_i p^{v_i}$ where p^{v_i} is the highest power of p dividing t_i, for $i = 1, \ldots, \ell-1$. Note since $\underline{g}' \neq 1$, $t_i > 0$. Let $f_{i,\lambda_i} \in A_{\lambda_i}$ have order p^{w_i}. Then $w_i > v_i$, and $\langle f_{i,\lambda_i}, B \rangle$ is isomorphic to $C_{p^{w_i}} \mathrm{wr}^{\Lambda} B$. Now $[f_{i,\lambda_i}, c_{i,1}, \ldots, c_{i,k_i}]$ is a product of conjugates of f_{i,λ_i} which commute, so if $f_{i,\lambda_{i+1}}^{t_i}$ is the power of $f_{i,\lambda_{i+1}}$ in (24), then (24) is at most a p^{v_i}th power. By Lemma 3.6, we thus have $k_i < c(C_{p^{v_{i+1}}} \mathrm{wr}^{\Lambda} B)$ for $i = 1, \ldots, \ell-1$, and so from (27) each h of type (21) in the product expressing g is such that

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \mathrm{wr}^{\Lambda} B) + 1 + m \geqslant w,$$

as required for (22). □

7.8 LEMMA

Let A be a p-group and let "Λ, B" be a faithful transitive pair such that B is a finite p-group, i.e. $\Lambda = (p^r)$ for some r in $\mathbb{Z}^+$. Identify A with the co-ordinate subgroup A_1 of A^{Λ}. Let $g, h \in A$, and for $\lambda \in (p^r) = \Lambda$ let g_λ, h_λ be the corresponding elements in A_λ. Let g have order p^w. Let $0 < t = up^v$ where p^v is the highest power of p dividing t and $v < w$. Then

$$\underline{g} = [g_\lambda^{t}, h_\lambda],$$

$$= [g_{\lambda'}^{u'}, b_1, \ldots, b_k, h_\lambda] \qquad \ldots\ldots.(28)$$

where $k = c(C_{p^{\nu+1}} \text{ wr}^{\Lambda} B) - 1$, $b_1, \dots, b_k \in B$, $\lambda' \in \Lambda$ and $u' \not\equiv 0 \mod p$. Furthermore we cannot find $k \geqslant c(C_{p^{\nu+1}} \text{ wr}^{\Lambda} B)$ to obtain a commutator of form (28) for $\underline{g}$.

<u>Proof</u>

Since $g_\lambda \in A_\lambda$ is of order p^w, $\langle g_\lambda, B \rangle \cong C_{p^w} \text{ wr}^{\Lambda} B$. Then by Lemma 3.6 there exists a non-trivial commutator of length $c(C_{p^{\nu+1}} \text{ wr}^{\Lambda} B)$ in $\langle g_\lambda, B \rangle$ which is a p^νth, but not a $p^{\nu+1}$th, power, and by Proposition 3.21, with λ as the chosen element rather than 1, we can take this to be of the form $[g_\lambda, c_1, \dots, c_k]$ where $k = c(C_{p^{\nu+1}} \text{ wr}^{\Lambda} B) - 1$, and $c_1, \dots, c_k \in B$. Now $[g_\lambda, c_1, \dots, c_k]$ is a product of conjugates of g_λ which commute, and so there exists λ_1 in Λ such that the power of g_{λ_1} in $[g_\lambda, c_1, \dots, c_k]$ is $g_{\lambda_1}^{\underline{u} p^\nu}$ where $\underline{u}$ is not divisible by p. Since B is transitive on Λ, there exists b in B such that $\lambda_1 b = \lambda$. Then since A_λ commutes with all $A_{\underline{\lambda}}$, $\lambda \neq \underline{\lambda}$,

$$[[g_\lambda, c_1, \dots, c_k]^b, h_\lambda] = [g_\lambda^{\underline{u} p^\nu}, h_\lambda].$$

But we want to obtain $[g_\lambda^{u p^\nu}, h_\lambda] = \underline{g}$ since $t = u p^\nu$ by definition. We can choose $u' \in \mathbb{Z}^+$ such that $u \equiv u' \underline{u} \mod p^w$. Since p is a prime and $u \not\equiv 0 \mod p$, we have $u', \underline{u} \not\equiv 0 \mod p$. Then

$$[[g_\lambda, c_1, \dots, c_k]^{u'b}, h_\lambda] = [g_\lambda^t, h_\lambda] = \underline{g}.$$

Now

$$[g_\lambda, c_1, \ldots, c_k]^{u'b} = [g_\lambda^{u'}, c_1, \ldots, c_k]^b \quad \text{by Lemma 3.17 iii),}$$
$$= [(g_\lambda^{b})^{u'}, c_1^{b}, \ldots, c_k^{b}],$$
$$= [g_{\lambda'}^{u'}, c_1^{b}, \ldots, c_k^{b}], \quad \text{where } \lambda' = \lambda b,$$
$$= [g_{\lambda'}^{u'}, b_1, \ldots, b_k],$$

where for $i = 1, \ldots, k$, $b_i = c_i^{b}$, and we have the result (28). Since by Lemma 3.6, if $k \geq c(C_{p^{v+1}} \operatorname{wr}^{\Lambda} B)$, any commutator of length $k+1$ is a p^{v+1} th power, by the above argument we cannot obtain (28) for such a k. □

7.9 LEMMA

Let $1 \neq \underline{h} = [[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell]$ be a simple pseudo-commutator in the p-group A, such that for $i = 1, \ldots, \ell-1$, $t_i = u_i p^{v_i}$, where p^{v_i} is the highest power of p dividing t_i. Let "Λ, B" be a faithful transitive pair such that B is a finite p-group, i.e. $\Lambda = (p^r)$ for some r. Identify A with the co-ordinate subgroup A_1 of A^{Λ}. Then we can rewrite $\underline{h}$ as a simple pseudo-commutator in $A \operatorname{wr}^{\Lambda} B$ of the form

$$\underline{h} = [\cdots[\cdots[h_{1,\lambda_1}^{u_1'}, c_{1,1}, \ldots, c_{1,k_1}, h_{2,\lambda_2}]^{u_2'}, c_{2,1}, \ldots, c_{2,k_2}, h_{3,\lambda_3}]^{u_3'}, \ldots, c_{\ell-1,k_{\ell-1}}, h_{\ell,\lambda_\ell}]$$

where for $i = 1, \ldots, \ell-1$, $\lambda_i \in \Lambda$, $h_{i,\lambda_i} \in A_{\lambda_i}$ is the conjugate of $h_i \in A_1$, $u_i' \not\equiv 0 \bmod p$, $k_i = c(C_{p^{v_i+1}} \operatorname{wr}^{\Lambda} B) - 1$, $c_{i,1}, \ldots, c_{i,k_i} \in B$, and $\lambda_\ell = 1$, so $h_{\ell,\lambda_\ell} = h_\ell$. Furthermore, no such expression exists if for some i in $\{1, \ldots, \ell-1\}$ we require $k_i \geq c(C_{p^{v_i+1}} \text{ wr}^{\Lambda} B)$.

Proof

We prove the result by an inverse induction on $\ell-1$. For $i = 1, \ldots, \ell-1$, let

$$g_{i,1} = [\,[\cdots[\,h_1^{t_1}, h_2\,]^{t_2}, \ldots, h_{i-1}\,]^{t_{i-1}}, h_i\,] \in A_1 ,$$

and for $\lambda \in \Lambda$ let $g_{i,\lambda}$ be the conjugate of $g_{i,1}$ in A_λ , which exists since B is transitive on Λ . Since $\underline{h} \neq 1$, the order of $g_{i,1}$ is at least p^{v_i+1} . In the inverse induction we show that at the i th stage

$$\underline{h} = [\cdots[\,g_{i,\lambda_i}^{u_i'}, c_{i,1}, \ldots, c_{i,k_i}, h_{i+1,\lambda_{i+1}}\,]^{u_{i+1}'}, c_{i+1,1}, \ldots, h_\ell\,] ,$$

and no such commutator exists for $k_i \geq c(C_{p^{v_i+1}} \text{ wr}^{\Lambda} B)$ . ...(29)

The base case is

$\ell-1$: This follows immediately from Lemma 7.8 .

Now suppose the result is true for $i+1$, i.e. we have shown

$$\underline{h} = [\cdots[\,g_{i+1,\lambda_{i+1}}^{u_{i+1}'}, c_{i+1,1}, \ldots, c_{i+1,k_{i+1}}, h_{i+2,\lambda_{i+2}}\,]^{u_{i+2}'}, \ldots, h_\ell\,] ,$$

and no such commutator exists for $k_{i+1} \geq c(C_{p^{v_{i+1}+1}} \text{ wr}^{\Lambda} B)$.

We need to show $g_{i+1,\lambda_{i+1}} = [\,g_{i,\lambda_i}^{u_i'}, c_{i,1}, \ldots, c_{i,k_i}, h_{i+1,\lambda_{i+1}}\,]$, for some $u_i' \not\equiv 0 \bmod p$, $k_i = c(C_{p^{v_i+1}} \text{ wr}^{\Lambda} B) - 1$, no such commutator exists for a larger k_i , and $c_{i,1}, \ldots, c_{i,k_i} \in B$, $\lambda_i \in \Lambda$.

Note that $g_{i+1,\lambda_{i+1}} = [\,g_{i,\lambda_{i+1}}^{t_i}, h_{i+1,\lambda_{i+1}}\,]$. The result (29) follows immediately from Lemma 7.8 , as required.

We thus have the result by induction. $\square$

7.10 REMARK

Lemma 7.9 tells us that given a non-trivial simple pseudo-commutator $\underline{h} = [[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell]$ in A , where for $i = 1, \ldots, \ell-1$, $t_i = u_i p^{v_i}$, such that p^{v_i} is the highest power of p dividing t_i , we can find a non-trivial simple pseudo-commutator in $A \text{ wr}^{\Lambda} B$ of $(1,0,1)$-length

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + c(C_{p^v} \text{ wr}^{\Lambda} B) \qquad \ldots\ldots\ldots(30)$$

where $\underline{h}$ has order p^v in A , and so (30) is a lower bound for $c(A \text{ wr}^{\Lambda} B)$ for each $\underline{h}$. Note we cannot find $b_1, \ldots, b_k \in B$, $k \geq c(C_{p^v} \text{ wr}^{\Lambda} B)$ such that $[\underline{h}, b_1, \ldots, b_k] \neq 1$ since $\langle \underline{h}, B \rangle$ is isomorphic to $C_{p^v} \text{ wr}^{\Lambda} B$. $\square$

7.11 COROLLARY

There exists a simple pseudo-commutator $1 \neq \underline{h}$ in A of the form given in Lemma 7.9 from which we can obtain a non-trivial simple pseudo-commutator of length $c = c(A \text{ wr}^{\Lambda} B)$ in $A \text{ wr}^{\Lambda} B$.

Proof

Let $g \in \gamma_c(A \text{ wr}^{\Lambda} B)$. Then since $c(A \text{ wr}^{\Lambda} B) > c(B)$ by Lemma 3.19 , $g \in A^{\Lambda}$, and so by Lemma 7.7 g is a product of non-trivial simple pseudo-commutators h of type (21) such that

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + 1 + m \geq c = c(A \text{ wr}^{\Lambda} B) , \qquad \ldots\ldots\ldots(31)$$

and if $[[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_\ell]$ is of order p^v

then by Lemma 7.9 we can find a non-trivial simple pseudo-commutator in $A \text{ wr}^{\Lambda} B$ of $(1,0,1)$-length

$$\sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + c(C_{p^v} \text{ wr}^{\Lambda} B) \qquad \ldots\ldots.(32)$$

where $c(C_{p^v} \text{ wr}^{\Lambda} B) - 1 \geq m$. Then $(32) \leq c$. Hence, on combining this with (31) ,

$$\begin{aligned} c &\leq \sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + 1 + m , \\ &\leq \sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + c(C_{p^v} \text{ wr}^{\Lambda} B) , \\ &\leq c , \end{aligned}$$

and we have the result. □

We make the following conjecture :

<u>7.12 CONJECTURE</u>

Let "Λ , B" be a faithful transitive pair and let B be a finite p-group. If

$$c(C_{p^2} \text{ wr}^{\Lambda} B) = c(C_p \text{ wr}^{\Lambda} B) + k ,$$

then

$$c(C_{p^n} \text{ wr}^{\Lambda} B) = c(C_p \text{ wr}^{\Lambda} B) + k(n-1) . \qquad □$$

This seems reasonable for three reasons :

1. By Corollary 3.1 ii) , if B is a finite p-group then

   $$c(C_{p^n} \text{ wr } B) = c(C_p \text{ wr } B) + (p-1)d(B)(n-1) ,$$

   i.e. the result holds for the standard wreath product which is a particular case of the permutational wreath product.

Note $k = (p-1)\,d(B)$.

2. If $B \leq P_2$ is of order p^t and exponent p then by Theorem 6.1 ,

$$c(C_{p^n}\ \mathrm{wr}^{(p^2)} B) = p + n(p-1)(t-1) ,$$

$$= c(C_p\ \mathrm{wr}^{(p^2)} B) + (p-1)(t-1)(n-1) .$$

By Theorem 6.2 , B is of maximal class $t-1$, and so since B is of exponent p , $d(B) = c(B) = t-1$. Hence $k = (p-1)\,d(B)$.

3. Let $B = C_{p^{n_2}}\ \mathrm{wr}\ C_{p^{n_3}}\ \mathrm{wr} \ldots \mathrm{wr}\ C_{p^{n_r}}$, where $n_i \geq 1$ for $i = 2, \ldots , r$, and let Λ be the set $(p^{n_2})\times(p^{n_3})\times \ldots \times(p^{n_r})$ on which B acts. Then by Theorem 3.4 , with $n_1 \geq 1$, and $m = \sum\limits_{i=2}^{r} n_i$,

$$c(C_{p^{n_1}}\ \mathrm{wr}^{\Lambda} B) = p^{m-1}\{\, p + (p-1)(n_1-1)\,\} ,$$

$$= p^m + p^{m-1}(p-1)(n_1-1) ,$$

$$= c(C_p\ \mathrm{wr}^{\Lambda} B) + p^{m-1}(p-1)(n_1-1) .$$

By Theorem 3.10 , $d(B) = p^{m-1}$, and so $k = (p-1)\,d(B)$.

In fact, if Conjecture 7.12 is true, we have

<u>7.13 COROLLARY</u>

The nilpotency class of $C_{p^n}\ \mathrm{wr}^{\Lambda} B$, where "Λ , B" is a faithful transitive pair and B is a finite p-group, is

$$c(C_p\ \mathrm{wr}^{\Lambda} B) + (p-1)\,d(B)\,(n-1) .$$

In other words, $k = (p-1)\,d(B)$, as in the cases 1, 2, and 3 above .

Proof

Recall from Corollary 1.3 that

$$d(B) = \max\{ w\, p^{s(w)-1} : 1 \leq w \leq c(B) \}$$

where a commutator of nilpotency weight w in B has order at most $p^{s(w)}$. Hence there exists g in B such that g is in $\gamma_u(B) \setminus \gamma_{u+1}(B)$, g has order p^s, and $up^{s-1} = d(B)$.

Now by the same argument as in the proof of Lemma 2.18, $C_{p^n} \operatorname{wr}^{\Lambda} B$ contains a subgroup isomorphic to $C_{p^n} \operatorname{wr} \langle g \rangle$, which in turn is isomorphic to $C_{p^n} \operatorname{wr} C_{p^s}$. By Corollary 2.30, there exists a non-trivial commutator h in $C_{p^n} \operatorname{wr} \langle g \rangle$ given by $h = [f_1, {}_{q-1}g]$, where $\langle f_1 \rangle = (C_{p^n})_1$, and $q = p^{s-1}\{ p + (p-1)(n-1) \}$. We may identify h with the commutator of the same form in $C_{p^n} \operatorname{wr}^{\Lambda} B$, which is thus also non-trivial. Now g is of nilpotency weight u, and so as a complex commutator in $C_{p^n} \operatorname{wr}^{\Lambda} B$, h has nilpotency weight at least

$$\begin{aligned} 1 + (q-1)u &= 1 + p^{s-1}\{ p + (p-1)(n-1) \} u - u, \\ &= 1 + d(B)\{ p + (p-1)(n-1) \} - u. \end{aligned}$$

Thus

$$\begin{aligned} c(C_{p^n} \operatorname{wr}^{\Lambda} B) &\geq 1 + d(B)\{ p + (p-1)(n-1) \} - u, \\ &\geq d(B)\{ p + (p-1)(n-1) \}. \qquad \ldots\ldots.(33) \end{aligned}$$

Furthermore, since "Λ, B" is a transitive pair, we can embed $C_{p^n} \operatorname{wr}^{\Lambda} B$ by Lemma 2.14 in $C_{p^n} \operatorname{wr} B$, and so

$$\begin{aligned} c(C_{p^n} \operatorname{wr}^{\Lambda} B) &\leq c(C_{p^n} \operatorname{wr} B), \\ &= a(B) + (p-1)d(B)(n-1) \quad \text{by Theorem 2.2}. \end{aligned}$$

$$\ldots\ldots.(34)$$

Thus if Conjecture 7.12 is true, by (33) we have

$$k(n-1) \geq d(B)\{p+(p-1)(n-1)\} - c(C_p \text{ wr}^{\Lambda} B),$$

and by (34) we have

$$k(n-1) \leq a(B) + (p-1)d(B)(n-1) - c(C_p \text{ wr}^{\Lambda} B).$$

Hence, dividing by $n-1$,

$$(p-1)d(B) + \frac{p\,d(B) - c(C_p \text{ wr}^{\Lambda} B)}{n-1} \leq k \leq (p-1)d(B) + \frac{a(B) - c(C_p \text{ wr}^{\Lambda} B)}{n-1}.$$

Let n tend to infinity. Then $k = (p-1)d(B)$ as required. □

7.14 DEFINITION

Let $a_{\Lambda}(B) = c(C_p \text{ wr}^{\Lambda} B)$ where "Λ, B" is a faithful transitive pair and B is a finite p-group. □

Note from (33) we have $a_{\Lambda}(B) > (p-1)d(B)$, and so the $(a_{\Lambda}(B),\underline{d}(B),1)$-series of A is defined. Let $\underline{h}$ be a non-trivial simple pseudo-commutator in A of the form

$$1 \neq \underline{h} = [[\cdots[h_1^{t_1}, h_2]^{t_2}, \ldots, h_{\ell-1}]^{t_{\ell-1}}, h_{\ell}]$$

where for $i = 1, \ldots, \ell-1$, $t_i = u_i p^{v_i}$, such that p^{v_i} is the highest power of p dividing t_i. Let $\underline{h}$ have order p^v. Then $\underline{h}^{p^{v-1}} \neq 1$ and has $(a_{\Lambda}(B),\underline{d}(B),1)$-length

$$a_{\Lambda}(B)\ell + \underline{d}(B)\{\sum_{i=1}^{\ell-1} v_i + v - 1\} \qquad \text{by Definition 7.2},$$

$$= \sum_{i=1}^{\ell-1} c(C_{p^{v_{i+1}}} \text{ wr}^{\Lambda} B) + c(C_{p^v} \text{ wr}^{\Lambda} B) \qquad \text{by Corollary 7.13},$$

which is (30). Thus if we assume Conjecture 7.12 is true, by Corollary 7.11 we have $c(A \text{ wr}^{\Lambda} B)$ is bounded above by the

$(a_\Lambda(B), \underline{d}(B), 1)$-class of A.

Now by Corollary 7.6, there exists $1 \neq g^{p^t}$ in A such that g is a simple commutator of length u in $\gamma_u(A) \setminus \gamma_{u+1}(A)$ and $a_\Lambda(B)u + \underline{d}(B)t = k$, where k is the $(a_\Lambda(B), \underline{d}(B), 1)$-class of A. Then by Remark 7.10, g gives rise to a non-trivial simple pseudo-commutator in $A \text{ wr}^\Lambda B$ of $(1,0,1)$-length equal to the $(a_\Lambda(B), \underline{d}(B), 1)$-length k of g^{p^t}. Hence $c(A \text{ wr}^\Lambda B)$ is bounded below by the $(a_\Lambda(B), \underline{d}(B), 1)$-class of A, and so

$$c(A \text{ wr}^\Lambda B) = (a_\Lambda(B), \underline{d}(B), 1)\text{-class of } A,$$

$$= \max\{a_\Lambda(B)w + \underline{d}(B)(s(w) - 1) : 1 \leq w \leq c(A)\}$$

where a commutator of nilpotency weight w in A has order at most $p^{s(w)}$, by Corollary 1.4.

In other words, Conjecture 3.2 ii) is true if Conjecture 7.12 is true.

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